Vector Valued Fourier Transforms and Absolutely Continuous Operators

Dissertation

zur Erlangung des akademischen Grades
doctor rerum naturalium (Dr. rer. nat.)

vorgelegt dem Rat der
Fakultät für Mathematik und Informatik
der Friedrich-Schiller-Universität Jena

von Dipl.-Math. Mariusz Piotrowski
geboren am 17. November 1977 in Czarne

Gutachter

1. PD Dr. A. Hinrichs
2. Prof. Dr. B. Carl
3. Prof. Dr. T. Kühn

Tag der letzten Prüfung des Rigorosums: 12.01.2005

Tag der öffentlichen Verteidigung: 28.01.2005

Acknowledgements

I would like to express my deepest appreciation to my teacher Professor Aicke Hinrichs for his great support and many stimulating hints and comments. Chapter 5 is a joint work with Aicke Hinrichs. I thank him for the permission to include our results here. I thank Professor B. Carl, Professor A. Hinrichs and Professor T. Kühn for kindly agreeing to serve as referee for this thesis. Further, I thank the following individuals for useful discussions and comments: Susanne Hergenhan, Iwona Piotrowska and Steffen Winter. In addition, I wish to thank the Deutsche Forschungsgemeinschaft (DFG) for financial support.

Contents

Zusammenfassung

Eine der wichtigsten Grundideen in der Banachraumtheorie der vergangenen Jahrzehnte ist es zu untersuchen, inwieweit sich Resultate und Methoden für skalarwertige Funktionen auf vektorwertige Funktionen verallgemeinern lassen. Der interessanteste Fall ergibt sich, wenn die untersuchte Verallgemeinerung von der Geometrie und der lokalen Struktur der beteiligten Räume und Operatoren abhängt. Eines der bekanntesten Beispiele hierfür ist sicherlich die vektorwertige Verallgemeinerung der Besselschen Ungleichung. Ein grundlegendes Resultat von S. Kwapień [Kwa72] besagt, daß ein Banachraum X genau dann isomorph zu einem Hilbertraum ist, wenn die Ungleichung

$$\left(\sum_{k=1}^{N} \left\| \int_0^1 f(t) e^{-2\pi i k t} \right\|^2 dt \right)^{1/2} \leq C \left(\int_0^1 \|f(t)\|^2 dt \right)^{1/2}$$

mit einer festen Konstante $C > 0$ für alle X-wertigen Funktionen f und $N = 1, 2, \ldots$ gilt. Das Hauptaugenmerk dieser Arbeit liegt auf den Beziehungen zwischen der Geometrie der Banachräume und den analytischen Eigenschaften der untersuchten Funktionen. Ein zweiter Schwerpunkt ist es, diejenigen Banachräume und Operatoren zu klassifizieren, für welche bestimmte Eigenschaften, die im skalaren Fall gelten, auch im vektorwertigen Fall wahr bleiben.

Die vorliegende Arbeit ist im wesentlichen in zwei Teile gegliedert. Im ersten Teil, der die Kapitel 2 bis 4 umfasst, beschäftigen wir uns mit der Fouriertyptheorie. Die Definition des Fouriertyps für Banachräume wurde 1969 von J. Peetre [Pee69] eingeführt. Er untersuchte die vektorwertigen Fouriertransformationen auf der reellen Achse. M. Milman setzte dieses Studium mit der Untersuchung von Fouriertransformationen bezüglich lokalkompakter abelscher Gruppen fort, siehe [Mil84]. Das oben erwähnte Resultat von S. Kwapień besagt gerade, daß Banachräume vom Fouriertyp 2 isomorph zu Hilberträumen sind. Das tiefliegendste Ergebnis dieser Theorie geht auf J. Bourgain zurück. Er bewies in [Bou82, Bou88], daß ein Banachraum einen nichttrivialen Fouriertyp bezüglich der klassichen Gruppen oder bezüglich der Cantorgruppe genau dann besitzt, wenn er B-konvex ist. Weitere Ergebnisse zur Hausdorff-Young-Ungleichung für vektorwertige Fouriertransformationen finden sich in den Arbeiten von M. E. Andersson [And98], J. Garcia-Cuerva, S. Kazarian, V.I. Kolyada and J.L. Torrea, [GKKT98], A. Pietsch, J. Wenzel [PW98] und A. Hinrichs [Hin01a, Hin01b, Hin03a].

Der zweite Teil dieser Arbeit besteht aus dem Kapitel 6 und stellt Resultate zur Absolutstetigkeit von Operatoren zwischen Banachräumen vor. Durch die Verallgemeinerung des Fakts, daß jeder schwach kompakte Operator auf einem Raum $C(\Omega)$ durch ein gewisses Integral bezüglich eines Kontrollmaßes abgeschätzt werden kann, führte C.P. Niculescu 1975 in [Nic75] ein Absolutstetigkeitskonzept für Operatoren zwischen Banachräumen ein. H. Jarchow und A. Pełczyński stellten in [Jar81] eine Beziehung zwischen dem Absolutstetigkeitsbegriff von Niculescu und der Operatorenidealteorie her. Ein systematisches Studium des Absolutstetigkeitskonzepts wurde in den Arbeiten von H. Jarchow, U. Matter [JM88, Mat87, Mat89] und F. Räbiger [Räb91] durchgeführt.

Wir gehen nun kurz auf den Inhalt der Arbeit ein. Eine ausführlichere Beschreibung findet sich am Anfang jedes einzelnen Kapitels. Kapitel 2 dient zur Festlegung der Notation und zur Bereitstellung von Hilfsmitteln. In Kapitel 3 zeigen wir, daß der Begriff des Fouriertyps p bezüglich der Cantorgruppe und ihres stetigen Analogons äquivalent sind. Der Beweis stützt sich wesentlich auf Techniken, die von J. Bourgain [Bou88] und H. König [Koe91] entwickelt wurden. In Kapitel 4 wird Bourgains Hausdorff-Young-Ungleichung verallgemeinert. Wir werden beweisen, daß die B-konvexen Räume gerade die Räume sind, die einen nichttrivialen Fouriertyp bezüglich des unendlichen Produktes der zyklischen Gruppen von Primzahlpotenzordnung haben. Dafür führen wir mittels des Orthonormalsystems von Vilenkin eine gewisse Klasse von Operatoren ein. In Kapitel 5 zeigen wir ein Übertragungsprinzip des Fouriertyps 2 vom klassischen Fall zum Fall einer beliebigen lokalkompakten abelschen Gruppe. Das Hauptresultat hier ist, daß unter allen Idealen von Fouriertyp-2-Operatoren das Ideal der Operatoren vom klassischen Fouriertyp 2 das kleinste ist. Es ist das Ergebnis einer gemeinsamen Arbeit mit A. Hinrichs, siehe dazu [HP03]. Im letzten Kapitel beschreiben wir eine Verallgemeinerung der von U. Matter [Mat87] eingeführten Prozedur, mittels der aus gegebenen Banachidealen $\mathcal{A}$ und $\mathcal{B}$ ein neues Banachideal $(\mathcal{A}, \mathcal{B})_\varphi$ definiert wird. Insbesondere stellen wir eine Verbindung zwischen bestimmten Interpolationsmethoden und der Absolutstetigkeit von Operatoren her, indem wir (p, φ)-absolutstetige Operatoren charakterisieren. Schließlich wenden wir dies auf die Abschätzung von Approximationsgrößen von Operatoren an.

Chapter 1

Introduction

It is an important theme in Banach space theory during the last decades to investigate how well results and methods for scalar valued functions can be extended to vector valued functions. The most interesting case occurs when the extension in question depends on geometry and structure of the underlying Banach spaces or operators. A prominent example illustrating this phenomenon is certainly the extension of Bessel's inequality to the vector valued setting. This is a celebrated result of S. Kwapień [Kwa72], which says that a Banach space X is isomorphic to a Hilbert space if and only if

$$\left(\sum_{k=1}^{N}\left\|\int_0^1 f(t)e^{-2\pi ikt}\right\|^2 dt\right)^{1/2} \leq C\left(\int_0^1 \|f(t)\|^2 dt\right)^{1/2}$$

holds with the constant $C > 0$ for every X-valued function f and $N = 1, 2, \ldots$. The basic idea of this thesis is to study the interaction between geometry of Banach spaces or operators and analytical properties of involved functions. Our second aim is to classify Banach spaces or operators for which certain property true in the scalar case remains valid in the vector valued case.

This thesis consists mainly of two parts. The first one comprises Chapters 2 through 4 and is devoted to the study of Fourier type theory. The investigation of the classical vector valued Fourier transform was initiated by J. Peetre in [Pee69]. M. Milman continued this research in [Mil84], where he considered vector valued Fourier transforms on general locally compact abelian groups. The above stated result of

S. Kwapień shows that Banach spaces of Fourier type 2 are isomorphic to Hilbert spaces. The deepest result in this direction is due to J. Bourgain [Bou82, Bou88]. He showed that a Banach space has nontrivial Fourier type with respect to the classical locally compact abelian groups or with respect to the Cantor group if and only if it is B-convex. A systematic study of vector valued Haussdorf-Young inequalities was initiated in the work of M. E. Andersson [And98], J. Garcia-Cuerva, S. Kazarian, V.I. Kolyada and J.L. Torrea, [GKKT98], A. Pietsch, J. Wenzel [PW98] and A. Hinrichs [Hin01a, Hin01b, Hin03a].

The second part of this thesis consists of Chapter 6 and deals with results on absolutely continuous operators. Taking into account special properties of weakly compact operators on $C(\Omega)$, see [BDS55], C.P. Niculescu introduced in [Nic75] the class of absolutely continuous operators. H. Jarchow and A. Pełczyński established in [Jar81] a result which connects the class of absolutely continuous operators with the theory of operator ideals. Further studies of this theme are due to H. Jarchow, U. Matter [JM88, Mat87, Mat89] and F. Räbiger [Räb91].

Let us now present the contents of this thesis in some detail. Chapter 2 collects fundamental notations and concepts. In Chapter 3 we show that the notions of Fourier type p with respect to the Cantor group and its continual analogue are equivalent. In the proof of this result techniques developed by J. Bourgain [Bou88] and H. König [Koe91] are used. In Chapter 4 we get an extension of Bourgain's Hausdorff-Young inequality which states that B-convex spaces are just the spaces which have non-trivial Fourier type p (i.e. $p > 1$) with respect to the infinite product of cyclic groups of order a prime power. To do this, we distinguish a certain class of operators with the help of Vilenkin orthonormal systems. In Chapter 5 we show a transference principle of Fourier type 2 from the classical case to the case of arbitrary locally compact abelian group. The main result of this section states that among all ideals of Fourier type 2 operators the ideal of Fourier type 2 operators with respect to classical groups is the smallest. This is a joint result with A. Hinrichs, see [HP03]. In the last chapter we present a generalization of a certain procedure introduced by U. Matter [Mat87] by which from given Banach ideals $\mathcal{A}$ and $\mathcal{B}$ a new scale of Banach ideals $(\mathcal{A}, \mathcal{B})_\varphi$ is generated. In particular we stress an important connection of our construction to interpolation theory. As an application we characterize (p, φ)-absolutely continuous operators by a special factorization property through a suitable interpolation space. The last section is devoted to give some applications to approximation quantities and entropy numbers.

Chapter 2

Preliminaries

2.1 Notation and Conventions

In this section we collect some needed notations, which remain fixed throughout this work. The symbol $\mathbb{K}$ stands as a synonym for the scalar field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. Furthermore, we put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The cardinality of a set I will be denoted by $|I|$ or $\mathrm{card}(I)$. Let X, Y be *Banach spaces* over $\mathbb{K}$. The norm of a Banach space X will usually be denoted by $\|\cdot\|$ or $\|\cdot\|_X$, $\|\cdot|X\|$ if more precision is desirable. The *closed unit ball* of X is given by $B_X = \{x \in X : \ \|x\| \leq 1\}$. The term *operator* means bounded linear operator unless specified otherwise. For an operator T from X into Y its *operator norm* is given by $\|T\| = \sup\{\|Tx\| : \ x \in B_X\}$. The set of all operators from X into Y is denoted by $\mathcal{L}(X, Y)$. If $X = Y$ then we simply write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$. We denote by I_X the identity map of X. The *dual space* of X is given by $X' := \mathcal{L}(X, \mathbb{K})$. Analogously we denote the *dual operator* of T by T'. The fact that the Banach space X embeds continuously into Y will always be denoted by $X \hookrightarrow Y$. For a real number a let $[a]$ represent the greatest integer less than or equal to a and $\{a\} = a - [a]$ the fractional part of a. The characteristic function of a measurable set A will be denoted by χ_A. Let (Ω, μ) be a σ-finite measure space. A function $f : \Omega \to X$ is called *μ-simple* if it can be written in the form

$$f = \sum_{k=1}^{n} x_k \chi_k,$$

where $x_1, \ldots, x_n \in X$ and $\chi_1, \ldots, \chi_n$ are characteristic functions of measurable subsets $A_1, \ldots, A_n$ of Ω. A function $f : \Omega \to X$ is said to be *strongly μ-measurable* if it coincides almost everywhere with the pointwise limit of simple X-valued functions. We denote by $L_p^X = L_p^X(\Omega, \mu)$ for $1 \leq p < \infty$ the *Bochner-Lebesgue space* of equivalence classes of strongly μ-measurable functions $f : \Omega \to X$ for which

$$\|f\|_p = \left(\int_\Omega \|f(t)\|_X^p \, d\mu(t) \right)^{1/p}$$

is finite. In the limiting case $p = \infty$ the usual modification with the essential supremum is required. In the scalar case $X = \mathbb{K}$ we will write $L_p(\Omega, \mu)$ or short $L_p(\Omega)$, L_p. Taking $\Omega = \mathbb{N}, \mathbb{Z}$ or $\Omega = \{1, \ldots, n\}$ and counting measure as μ produces vector-valued sequence spaces denoted as usual by $\ell_p(X)$ and $\ell_p^n(X)$, respectively. It is known that for $1 \leq p < \infty$ the space $L_{p'}(\Omega, \mu)$ is isometrically isomorphic to $L_p(\Omega, \mu)'$. Here p' denotes the *conjugate exponent* of p given by $1/p + 1/p' = 1$. We denote by $(\bigoplus_{n=1}^\infty X_n)_p$ the closure of all finite sequences (x_n) with $x_n \in X_n$ for all $n \in \mathbb{N}$ endowed with the norm

$$\|(x_n)\| = \left(\sum_{n=1}^\infty \|x_n\|_{X_n}^p \right)^{1/p}.$$

The spaces $(\bigoplus_{i=1}^n X_i)_p$ are defined accordingly. In the sequel, we will also consider the subsequent generalization of ℓ_p-spaces. Given any sequence $x = (x_n)$ in a Banach space X with $\|x_n\| \to 0$ we arrange the norms of the members of this sequence in non-increasing order. The resulting sequence (x_n^*) will be called the *non-increasing rearrangement*. The *Lorentz sequence space* $\ell_{p,q}(X)$ (or $\ell_{p,q}^n(X)$ for finite sequences) with $q < \infty$ consists of all sequences (x_n) equipped with the norm

$$\|(x_n)\|_{p,q} = \left(\sum_n \left(n^{1/p-1} \|x_n^*\| \right)^q \right)^{1/q}.$$

The sequence space $\ell_{p,\infty}$ (weak ℓ_p) is equipped with the quasi-norm

$$\|(x_n)\|_{p,\infty} = \sup n^{1/p} \|x_n^*\|.$$

The spaces obtained in this way are Banach spaces though $\| \cdot \|_{p,q}$ is, in general, only equivalent to a norm. Let $1 < r < s < \infty$. We will also work with the *Lorentz* *quasi-norm* given by

$$\|(x_k)|\ell_{r,1}^n\|_s = \min\Big\{\sum_{h=1}^{m} |\mathbb{F}_h|^{1/r-1/s}\Big(\sum_{k\in\mathbb{F}_h}\|x_k\|^s\Big)^{1/s}\Big\},$$

where the minimum ranges over all partitions of $\{1,\dots n\}$ into pairwise disjoint subsets $\mathbb{F}_h$ with $h = 1,\dots,m$ and $m \le n$. Below, we summarize Lorentz quasi-norm estimates which we will need later on.

LEMMA 2.1. *Let $1 < r < s < \infty$. Then there are constants $A, B \ge 1$ depending only on r and s such that*

$$\|\cdot|\ell_{r,1}^n\| \le An^{1/r-1/s}\|\cdot|\ell_s^n\| \tag{2.1}$$

$$1/2\,\|\cdot|\ell_{r,1}^n\|_s \le \|\cdot|\ell_{r,1}^n\| \le B\|\cdot|\ell_{r,1}^n\|_s. \tag{2.2}$$

We omit the proof of this lemma. It may be found in [PW98, Section 0.5].

2.2 Operator ideals and ideal norms

In this section we collect some notation and results from the theory of operator ideals. For more information we refer the reader to the monograph [Pie78].
Recall that $\mathcal{L}$ denotes the class of all operators between Banach spaces. Moreover, the class of all Banach spaces is denoted by L. An *operator ideal* $\mathcal{A}$ is a subclass of $\mathcal{L}$ such that the components $\mathcal{A}(X,Y) = \mathcal{A}\cap\mathcal{L}(X,Y)$ are linear spaces which satisfy the *ideal property*

$$UTV \in \mathcal{A}(X_0,Y_0) \ \text{ for } \ V \in \mathcal{L}(X_0,X), T \in \mathcal{A}(X,Y) \ \text{ and } \ U \in \mathcal{L}(Y,Y_0). \tag{2.3}$$

With every operator ideal $\mathcal{A}$ we associate the class of all Banach spaces X such that the identity map I_X belongs to that ideal, i.e.

$$\mathsf{A} = \{X \in \mathsf{L} : I_X \in \mathcal{A}\}.$$

An *ideal (quasi-)norm* α is a function on $\mathcal{L}$ which is a (quasi-)norm and satisfies on each component $\mathcal{L}(X,Y)$ the *ideal norm property*

$$\alpha(UTV) \le \|U\|\alpha(T)\|V\| \ \text{ for } \ V \in \mathcal{L}(X_0,X), T \in \mathcal{A}(X,Y) \ \text{ and } \ U \in \mathcal{L}(Y,Y_0). \tag{2.4}$$

We will write $\alpha(X)$ for $\alpha(I_X)$. It is well known that every ideal norm is equivalent to the operator norm.

An *ideal (quasi-) norm* $\|\cdot|\mathcal{A}\|$ on an operator ideal $\mathcal{A}$ is a function on $\mathcal{A}$ which is a (quasi-) norm and satisfies on each component $\mathcal{A}(X,Y)$ the *ideal norm property*

$$\|UTV|\mathcal{A}\| \le \|U\|\|T|\mathcal{A}\|\|V\|.$$

Observe that (2.4) and the above notion coincide for $\mathcal{A} = \mathcal{L}$. An operator ideal $\mathcal{A}$ endowed with an ideal (quasi-) norm such that all components $\mathcal{A}(X,Y)$ are complete will be called *(quasi) Banach operator ideal*.

The sequence (α_n) of ideal norms will be called *submultiplicative* if $\alpha_{mn}(UT) \le \alpha_m(U)\alpha_n(T)$ for all $m,n \in \mathbb{N}$, which we abbreviate by $\alpha_{mn} \le \alpha_m \circ \alpha_n$. Given any sequence of ideal norms (α_n) we put

$$\mathcal{L}[\alpha_n] = \{T \in \mathcal{L} :\ (\alpha_n(T)) \text{ is bounded}\}$$

and

$$\mathcal{L}_0[\alpha_n] = \{T \in \mathcal{L} :\ \alpha_n(T) \to 0 \text{ as } n \to \infty\}.$$

Then $\mathcal{L}[\alpha_n]$ and $\mathcal{L}_0[\alpha_n]$ endowed with the ideal norm $\sup_n \alpha_n(T)$ induced by $\mathcal{L}[\alpha_n]$ become Banach operator ideals. To avoid trivial cases we use instead of the sequence (α_n) a sequence $\lambda_n^{-1}\alpha_n$ with appropriate weights satisfying $\min(\alpha_n) \le \lambda_n \le \max(\alpha_n)$.

We now describe a few methods to produce new operator ideals from given ones. The *dual ideal* $\mathcal{A}^{\mathrm{dual}}$ consists of all operators T such that T' belongs to the given ideal $\mathcal{A}$. We say that $\mathcal{A}$ is *symmetric* if $\mathcal{A} = \mathcal{A}^{\mathrm{dual}}$. The *closed hull* $\overline{\mathcal{A}}$ of an ideal $\mathcal{A}$ has the components $\overline{\mathcal{A}(X,Y)}$, where the closure is taken with respect to the operator norm topology on $\mathcal{L}(X,Y)$. The ideals $\mathcal{A}$ such that $\mathcal{A} = \overline{\mathcal{A}}$ are said to be *closed*.

The *injective hull* $\mathcal{A}^{\mathrm{inj}}$ consists of all operators $T \in \mathcal{L}(X,Y)$ that become a member of $\mathcal{A}$ by extending the codomain $X \xrightarrow{T} Y \xrightarrow{J} Y_0$. Here J denotes an injection into a suitable Banach space Y_0. Due to the extension property we may take $Y_0 = \ell_\infty(I)$ for an appropriate index set I. An ideal is called *injective* if $\mathcal{A} = \mathcal{A}^{\mathrm{inj}}$. Injectivity of $\mathcal{A}$ implies that the associated class of Banach spaces denoted by A is stable when passing to subspaces.

2.3 Abstract harmonic analysis and Fourier type theory

This section reviews some of the standard facts on abstract harmonic analysis and Fourier type theory. For more information, we recommend the monographs [HR79] and [PW98].

We will work in the framework of a *topological group* G, i.e. a group equipped with a topology with respect to which the group operations are continuous. Any topological group mentioned in this section will be taken to be an abelian group and a locally compact (or sometimes simply compact) Hausdorff space. For simplicity let us use the abbreviation lca for "locally compact abelian". An isomorphism of the topological groups, denoted by $\cong$, is a map which is both a group isomorphism and a homeomorphism. It is well known that on every lca group there exists a translation invariant Borel measure μ_G, the so-called *Haar measure*. For a construction of such a measure we refer the reader to [HR79]. In the case when G is compact, it is customary to normalize μ_G by declaring that $\mu_G(G) = 1$. An exeption are finite groups where also the counting measure is used. It will be clear from the context which convention we use in this case. For lca groups the uniqueness of the Haar measure holds up to a positive constant.

Let now G be a lca group. A *character* of G is a continuous group homomorphism $\gamma : G \to \mathbb{T}$ to the unit torus $\mathbb{T} = \{z \in \mathbb{C} : \; |z| = 1\}$, that is γ is a complex-valued map on G satisfying

$$|\gamma(t)| = 1 \text{ and } \; \gamma(st) = \gamma(s)\gamma(t)$$

for all $s, t \in G$. The collection of all characters on G is an abelian group under pointwise multiplication and carries a natural locally compact topology. The resulting lca group is the *dual group* G' of G. The *Fourier transform* $\mathcal{F}_G$ of a function $f \in L_1(G)$ is given by

$$\mathcal{F}_G(f)(\gamma) = \int_G f(t)\overline{\gamma(t)}\, d\mu_G(t) \quad \text{for} \quad \gamma \in G'.$$

It is a function in $C_0(G')$, the space of continuous functions on G' vanishing at infinity. To simplify notation, we will write $\widehat{f}$ for $\mathcal{F}_G f$ if it is clear which group is considered. In the sequel the following groups will be called "classical".

EXAMPLES.

- The integers $\mathbb{Z}$ with addition, discrete topology and counting measure. The characters on $\mathbb{Z}$ are given by $\gamma(k) = z^k$ for some $z \in \mathbb{T}$. It turns out that $\mathbb{Z}' \cong \mathbb{T}$ and the Fourier transform is given by

$$\widehat{f}(e^{it}) = \sum_{n\in\mathbb{Z}} f(n)e^{-int} \quad \text{for} \quad e^{it} \in \mathbb{T}.$$

- The torus $\mathbb{T}$ with multiplication, usual topology and normalized Lebesgue measure. The characters on $\mathbb{T}$ are given by $\gamma(z) = z^k$ for some $k \in \mathbb{Z}$. It turns out that $\mathbb{T}' \cong \mathbb{Z}$ and the Fourier transform is given by

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-int}\,dt \quad \text{for} \quad n \in \mathbb{Z}.$$

- The real line $\mathbb{R}$ with multiplication, usual topology and Lebesgue measure. The characters on $\mathbb{R}$ are given by $\gamma(x) = e^{ixy}$ with $y \in \mathbb{R}$. It turns out that $\mathbb{R}' \cong \mathbb{R}$ and the Fourier transform is given by

$$\widehat{f}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ixy}\,dx \quad \text{for} \quad y \in \mathbb{R}.$$

For more examples we refer the reader to Chapter 3, where we describe in detail the harmonic analysis on cyclic groups. The above examples show that $\mathbb{Z}$ and $\mathbb{T}$ are duals of each other. This is a special case of the following duality principle.

THEOREM 2.2 (Pontryagin)**.** *Let G be a lca group. The map*

$$\Phi : G \to G'', \quad \Phi(t)(\gamma) := \gamma(t)$$

is an isomorphism of the topological groups G and G''.

In particular it follows that the dual of a discrete group is a compact abelian group and conversely the dual of a compact group is a discrete abelian group. Furthermore, we may easily extend the classical Plancherel theorem to obtain that the Fourier transform $\mathcal{F}_G$ on $L_1(G) \cap L_2(G)$ can be extended to an isometric isomorphism from $L_2(G)$ to $L_2(G')$. The classical Hausdorff-Young inequality for G asserts that the Fourier transform extends to functions in $L_p(G)$ for $1 < p \leq 2$ so that $\mathcal{F}_G$ defines a bounded operator from $L_p(G)$ into $L_{p'}(G')$. It was computed by K.I. Babenko [Bab61] and W. Beckner [Bec75] that the exact value of $\|\mathcal{F} : L_p(\mathbb{R}) \to L_{p'}(\mathbb{R})\|$ is equal to $\sqrt{p^{1/p}/p'^{1/p'}}$.

From now on let p be a number from the interval $[1, 2]$. Due to the linearity of the Fourier transform, we may extend it to functions from the algebraic tensor product $L_p(G) \otimes X$ consisting of finite sums $f = \sum \varphi_k x_k$ with $\varphi_k \in L_p(G)$ and $x_k \in X$ by putting

$$\mathcal{F}_G f = \sum (\mathcal{F}_G \varphi) x_k \in L_{p'}(G) \otimes X.$$

In view of the density of $L_p(G) \otimes X$ in $L_p^X(G)$ it is natural to ask whether $\mathcal{F}_G$ extends to a bounded linear operator from $L_p^X(G)$ into $L_{p'}^X(G')$. This question gives rise to the following definition. We say that the Banach space X has *Fourier type p with respect to G* if the Fourier transform $\mathcal{F}_G$ extends to a bounded linear operator from $L_p^X(G)$ into $L_{p'}^X(G')$. In the language of inequalities it means that the Hausdorff-Young inequality

$$\|\mathcal{F}_G f\|_{L_{p'}^X(G')} \le C \|f\|_{L_p^X(G)} \tag{2.5}$$

holds for every X-valued function $f \in L_p^X(G)$.
That the extension in question does not exist in general is shown by the following simple example.

EXAMPLE. Put $G = \mathbb{Z}, G' = \mathbb{T}$ and $X = L_\infty(\mathbb{T})$. Furthermore, denote by (e_k) the unit vector basis in $\ell_p(\mathbb{Z})$. For a fixed integer N we define the function $f_N \in \ell_p(\mathbb{Z}) \otimes X$ by

$$f_N = \sum_{k=-N}^{N} e_k \otimes \exp(2\pi i k \cdot).$$

Considering f_N as a function in $\ell_p^X(\mathbb{Z})$ and $\widehat{f}_N$ as a function in $L_{p'}^X([0, 1))$ yields

$$\|f_N\|_p = \Big(\sum_{n \in \mathbb{Z}} \|f_N(n)\|_X^p \Big)^{1/p} = \Big(\sum_{k=-N}^{N} \|\exp(2\pi i k \cdot)\|_\infty^p \Big)^{1/p} = (2N+1)^{1/p}$$

and

$$\|\widehat{f}_N\|_{p'} = \Big(\int_0^1 \|\widehat{f}_N(s)\|^{p'} ds \Big)^{1/p'} = \Big(\int_0^1 \sum_{k=-N}^{N} \|\exp(2\pi i k(\cdot - s)\|_\infty^{p'} ds \Big)^{1/p'}$$

$$= 2N + 1.$$

Thus, for $p > 1$ it is not possible to find a constant $c > 0$ independent of N such that $\|\widehat{f}_N\|_{p'} \leq c\|f_N\|_p$, which shows that a bounded extension of the Fourier transform from $L_p^X(G)$ to $L_{p'}^X(G')$ does not exist.

This notion in the case of the classical Fourier transform on $\mathbb{R}$ was considered for the first time by Peetre in [Pee69]. The above notion can be extended to the case of operators in the following way. The operator T is said to have *Fourier type p* with respect to the group G if the operator

$$\mathcal{F}_G \otimes T : L_p(G) \otimes X \to L_{p'}(G') \otimes Y$$

admits a bounded and linear extension acting between $L_p^X(G)$ and $L_{p'}^Y(G')$. In other words we require the validity of a Hausdorff-Young inequality

$$\|(\mathcal{F}_G \otimes T)f\|_{L_{p'}^Y(G')} \leq C\|f\|_{L_p^X(G)} \tag{2.6}$$

for every X-valued function $f \in L_p^X(G)$. Recall that p' denotes the conjugate exponent of p defined by $1/p+1/p' = 1$. In particular, X has Fourier type p if the identity operator I_X does. The operator norm of the extended operator will be denoted by $\|T|\mathcal{FT}_p^G\|$. Trivially, any operator is of Fourier type 1 with respect to an arbitrary lca group. Moreover, by definition Fourier type p passes to subspaces. The class of all operators of Fourier type p with respect to the group G equipped with the norm $\| \cdot |\mathcal{FT}_p^G\|$ constitutes a Banach operator ideal denoted by $\mathcal{FT}_p^G$, see Section 2.2.

Observe that the ideals $\mathcal{FT}_p^G$ become more restrictive as p increases. More precisely, for $1 < p_1 < p_2 < 2$ we have

$$\mathcal{FT}_2^G \subset \mathcal{FT}_{p_2}^G \subset \mathcal{FT}_{p_1}^G \subset \mathcal{FT}_1^G = \mathcal{L}.$$

This was proved by M. Milman [Mil84]. Moreover, for any infinite group G all of the above inclusions are strict. It is shown in [GKKT98] that $L_{p_1} \in \mathcal{FT}_{p_1}^G \setminus \mathcal{FT}_{p_2}^G$.

A basic question arising from the above consideration is whether the ideals $\mathcal{FT}_p^G$ depend on the infinite lca group G? In other words, does there exist for given infinite lca groups G_1 and G_2, an operator T which has Fourier type p with respect to G_1 but not with respect to G_2? To be more precise, we are interested in transference principles like $\mathcal{FT}_p^{G_1} \subset \mathcal{FT}_p^{G_2}$ or $\mathcal{FT}_p^{G_1} = \mathcal{FT}_p^{G_2}$. The most prominent result in this direction states that for any $n \in \mathbb{N}$

$$\mathcal{FT}_p^{\mathbb{R}} = \mathcal{FT}_p^{\mathbb{T}} = \mathcal{FT}_p^{\mathbb{Z}} = \mathcal{FT}_p^{\mathbb{R}^n} = \mathcal{FT}_p^{\mathbb{T}^n} = \mathcal{FT}_p^{\mathbb{Z}^n}. \tag{2.7}$$

This was independently shown by Andersson [And98], König [Koe91] and Garcia-Cuerva et al. [GKKT98]. In [And98] Andersson obtained the following abstract results concerning subgroups and quotient groups. We remark that in the above cited papers only the case of identity operators of Banach spaces is treated. Nevertheless, the extension of the proofs to the operator case requires only straightforward modifications.

(i) If G_2 is an open subgroup of G_1, then $\mathcal{FT}_p^{G_1} \subset \mathcal{FT}_p^{G_2}$.

(ii) If G_2 is a compact subgroup of G_1, then $\mathcal{FT}_p^{G_1} \subset \mathcal{FT}_p^{G_1/G_2}$.

Next we want to look at the smallest class in the scale of Fourier type p operator ideals, $\mathcal{FT}_2^G$. By the result of S. Kwapień [Kwa72], a Banach space has Fourier type 2 if and only if it is isomorphic to a Hilbert space. A characterization like this in the operator case is still an open question. For more information and results on this problem we refer the reader to Chapter 5 where we prove that among all ideals of Fourier type 2 operators the ideal of Fourier type 2 operators with respect to the classical groups is the smallest. Let us now make some comments on the deepest result in the Fourier type theory achieved by J. Bourgain in [Bou82] and [Bou88] which states that B-convex spaces are just spaces of nontrivial Fourier type p (i.e. $p > 1$) with respect to the classical groups or the Cantor group. B-convex Banach spaces are spaces of nontrivial Rademacher type. By a result of G. Pisier [Pis73] this class coincides with the class of Banach spaces which do not contain the spaces ℓ_1^n uniformly. A general reference here is [DJT95]. For a treatment of a more general case we refer the reader to Chapter 4 where we prove Bourgain's result in the case of infinite products of cyclic groups of order a prime power.
We conclude this chapter by quoting some results on duality of Fourier type. The Pontryagin duality for lca groups and Banach space duality fit together very well to produce the following duality theorem, see [And98] or [GKKT98].

THEOREM 2.3. *An operator T has Fourier type p with respect to the lca group G if and only if the dual operator T' has Fourier type p with respect to the dual group G'. Moreover, we obtain $\|T|\mathcal{FT}_p^G\| = \|T'|\mathcal{FT}_p^{G'}\|$.*

However, it can easily be seen that in some cases even more is true. Indeed, we have $\mathcal{FT}_p^G = \mathcal{FT}_p^{G'}$ for the classical groups and the Cantor group. It was asked in [GKKT98], whether the equality in question holds for arbitrary lca group G. In [HL02] A. Hinrichs and H.H. Lee gave an affirmative answer to this question by showing that the operator ideal $\mathcal{FT}_p^G$ is symmetric.

Theory of Fourier type with respect to non-abelian compact groups was investigated in [GP02]. For applications of classical Fourier type to the study of abstract Cauchy problems and vector valued Fourier multipliers, we refer the reader to [GW01].

Chapter 3

Equivalence of Fourier-type p with respect to the Cantor group and its continual analogue

3.1 Harmonic analysis on the Cantor group

In this section, we describe basic facts of harmonic analysis on the Cantor group. For $n \in \mathbb{N}_0$ the *binary expansion* of n is $n = \sum_{k=0}^{\infty} n_k 2^k$ with its *binary coefficients* $n_k \in \{0, 1\}$. In fact we observe that the sum is a finite sum. For a pair of integers $n, m \in \mathbb{N}_0$ we define their *dyadic sum* by

$$n \oplus m := \sum_{k=0}^{\infty} |n_k - m_k| 2^k.$$

Here the sequences $(n_k)_{k \in \mathbb{N}}$ and $(m_k)_{k \in \mathbb{N}}$ denote the binary coefficients of n and m, respectively. We are now in the position to introduce the *Cantor group* (or dyadic group). The set

$$\mathbb{D} := \mathbb{Z}_2^{\infty} = \{x = (x_n)_{n \in \mathbb{N}} : x_n \in \{0, 1\}\}.$$

endowed with the addition given by $x + y := (|x_n - y_n|)_{n \in \mathbb{N}}$ becomes an abelian group. This group will be called the *Cantor group*. Moreover we equip this group with the direct product topology. We conclude from the Tichonov theorem that the Cantor group is compact. The Haar measure μ on $\mathbb{D}$ is the product measure of the measures that on each factor assign measure 1/2 to each of the two points 0 or 1. Observe that the subset given by $\mathbb{D}_0 := \{x \in \mathbb{D} : x_n \to 0 \quad \text{as} \quad n \to \infty\}$ is a countable subgroup of $\mathbb{D}$. It is known that the character group $\mathbb{D}'$ of $\mathbb{D}$ is isomorphic to $(\mathbb{N}_0, \oplus)$. We have the following formula for the characters on $\mathbb{D}$.

$$\psi_n(x) = (-1)^{\langle n,x \rangle},$$

where

$$\langle n, x \rangle := n_0 x_0 + n_1 x_1 + \ldots \pmod 2$$

for $n \in \mathbb{N}$ and $x \in \mathbb{D}$. For $n \in \mathbb{N}_0$ the *Fourier transform with respect to the Cantor group* of a function $f \in L_1^X(\mathbb{D})$ is defined by

$$(\mathcal{F}_{\mathbb{D}} f)(n) = \widehat{f}(n) = \int_{\mathbb{D}} f \psi_n d\mu, \tag{3.1}$$

where the integral is interpreted in the sense of Bochner and μ is the Haar measure on $\mathbb{D}$. For convenience we identify the Cantor group with the interval $[0, 1]$. More precisely, the map

$$\Phi : \mathbb{D} \to [0, 1], \quad (x_k) \mapsto \sum_{k=0}^{\infty} x_k 2^{-k-1} \tag{3.2}$$

establishes a one-to-one correspondence between non-periodic elements of $\mathbb{D}$ and binary irrational points $t \in [0, 1]$. Moreover, Φ is measurable and maps the Haar measure μ on $\mathbb{D}$ to the Lebesgue measure λ on $[0, 1]$. This enables us to introduce a more convenient description of the character group of the Cantor group which provides a useful enumeration of characters. For the dyadic expansion of a positive integer $n = \sum_{k=0}^{\infty} n_k 2^k$ with $n_k \in \{0, 1\}$, the n-th *Walsh function* is given by

$$w_n(t) = \prod_{k=0}^{\infty} (r_k(t))^{n_k}. \tag{3.3}$$

Here r_k stands for the k-th *Rademacher function* defined by $r_k(t) = \text{sign}(\sin 2^k \pi t)$. Observe that the infinite product on the right hand side is again finite, hence w_n is well defined. The sequence of Walsh functions constitutes the orthonormal *Walsh system*.

An alternative formula for the Fourier transform of an integrable function defined on $\mathbb{D}$ reads as follows

$$\widehat{f}(n) = \int_0^1 f(t)w_n(t)d\lambda(t) \quad \text{for} \quad n \in \mathbb{N}_0 \quad \text{and} \quad t \in [0,1].$$

We now shall construct a continual analogue of the Cantor group. Let us consider the set of doubly infinite sequences

$$\mathbb{F} = \{x = (x_n)_{n\in\mathbb{Z}} : x_n = 0 \quad \text{or } 1 \quad \text{and } x_n \to 0 \quad \text{as } n \to -\infty\}.$$

The sum operation on $\mathbb{F}$ is defined analogous as in the case of the Cantor group. The product of $x = (x_n)_{n\in\mathbb{Z}}$ and $y = (y_n)_{n\in\mathbb{Z}}$ is given by

$$x \cdot y := (\xi_n)_{n\in\mathbb{Z}} \quad \text{with} \quad \xi_n = \sum_{i+j=n} x_i y_j \pmod 2.$$

Equipped with the discrete product topology the group $\mathbb{F}$ becomes a locally compact abelian group. Using the mapping

$$x \mapsto \sum_{n\in\mathbb{Z}} x_n 2^{-n-1} \tag{3.4}$$

we identify the group $\mathbb{F}$ with the positive real axis $\mathbb{R}_+ = [0,\infty)$. In the following we carefully distinguish between the dyadic multiplication $x \cdot y$ and the usual multiplication xy. We generate the characters on $(\mathbb{F},+)$ as follows. For $x, y \in \mathbb{F}$ define

$$\psi_y(x) := (-1)^{\pi_{-1}(x\cdot y)},$$

where $\pi_n : \mathbb{F} \to \{0,1\}$ is given by $\pi_n(x) = \pi_n((x_j)_{j\in\mathbb{Z}}) := x_n$. It is clear that

$$\psi_y(x) = \psi_x(y) \tag{3.5}$$

and

$$\psi_y(x) = \psi_{[y]}(x)\psi_{[x]}(y) \tag{3.6}$$

We consider the *Fourier transform with respect to the group* $\mathbb{F}$ of a function $f \in L_1^X(\mathbb{F})$ defined by

$$\mathcal{F}_{\mathbb{F}} f(y) = \int_{\mathbb{F}} f(x)\psi_y(x)d\mu(x) \quad \text{for} \quad y \in \mathbb{F}.$$

Here, as usual, μ denotes the up to a multiplicative positive constant unique Haar measure on $\mathbb{F}$ and the integral is understood in the sense of Bochner. Due to the formula (3.4) we may write equivalently

$$\mathcal{F}_{\mathbb{F}} f(y) = \int_0^\infty f(x)\psi_y(x)dx \quad \text{for} \quad y \in \mathbb{R}_+.$$

For more information on dyadic harmonic analysis the reader is referred to [SWS90] and [GES91].

3.2 Result

From now on, p will always be in the interval $[1, 2]$. According to the definition from Section 2.2 we introduce the class of Banach spaces of Fourier type p with respect to the Cantor group and its continual counterpart in the following way.

(i) A Banach space X has Fourier type p with respect to the group $\mathbb{D}^m$ if there is a constant $C > 0$ such that the Hausdorff-Young inequality

$$\left(\sum_{n\in\mathbb{N}_0^m} \left\|(\mathcal{F}_{\mathbb{D}^m} f)(n)\right\|^{p'} \right)^{1/p'} \leq C \left(\int_{[0,1]^m} \|f(t)\|^p dt \right)^{1/p}$$

holds for any function $f \in L_p^X(\mathbb{D}^m)$.

(ii) A Banach space X has Fourier type p with respect to the group $\mathbb{F}^m$ if there is a constant $C > 0$ such that the Hausdorff-Young inequality

$$\left(\int_{\mathbb{R}_+^m} \left\|(\mathcal{F}_{\mathbb{F}^m} f)(t)\right\|^{p'} dt \right)^{1/p'} \leq C \left(\int_{\mathbb{R}_+^m} \|f(t)\|^p dt \right)^{1/p}$$

holds for any function $f \in L_p^X(\mathbb{F}^m)$.

We are now ready to state the main result of this chapter which says that the notions of Fourier type p with respect to the Cantor group and its continual counterpart coincide.

THEOREM 3.1. *Let $1 < p < 2$. For a Banach space X the following statements are equivalent* *(i) X has Fourier type p with respect to group $\mathbb{D}$.*

(ii) X has Fourier type p with respect to group $\mathbb{D}^m$ for all $m \in \mathbb{N}$.

(iii) X has Fourier type p with respect to group $\mathbb{F}$.

(iv) X has Fourier type p with respect to group $\mathbb{F}^m$ for all $m \in \mathbb{N}$.

Moreover, in this case all norms coincide.

Proof. Our proof is based upon ideas found in [Bou88] and [Koe91]. First we observe that by the definitions of the underlying groups the equivalences $(i) \Leftrightarrow (ii)$ and $(iii) \Leftrightarrow (iv)$ together with the equality of the corresponding norms are obvious. Let us start with the implication $(i) \Rightarrow (iii)$. So assume that X has Fourier type p with respect to the group $\mathbb{D}$, i.e. for any function $g \in L_p^X(\mathbb{D})$ we have the inequality

$$\left(\sum_{n\in\mathbb{N}} \|\widehat{g}(n)\|_X^{p'} \right)^{1/p'} \leq C\|g\|_p. \tag{3.7}$$

Let $f \in L_p^X(\mathbb{R}_+)$ be supported in the interval $[0,1]$. Our goal is to prove that $\|\mathcal{F}_{\mathbb{F}} f\|_{p'} \leq C\|f\|_p$. Using the correspondence (3.2) we identify the group $\mathbb{D}$ with the interval $[0,1]$ and consider the function g on $[0,1]$ given by

$$g(t) = f(t)\psi_y(t).$$

From (3.5) we obtain

$$\widehat{g}(n) = \int_0^1 f(t)\psi_y(t)\psi_n(t)dt = \int_0^\infty f(t)\psi_t(n+y)dt = \mathcal{F}_{\mathbb{F}}(n+y).$$

Since $\|f\|_p = \|g\|_p$, it follows from (3.7) that

$$\left(\sum_{n\in\mathbb{N}} \|\mathcal{F}_{\mathbb{F}}(n+y)\|_X^{p'} \right)^{1/p'} \leq C\|f\|_{L_p^X(\mathbb{R}_+)}.$$

Integration of the last inequality over $y \in [0,1]$ gives the claim. In general, for compactly supported $f \in L_p^X(\mathbb{R}_+)$ we consider the function $f_\varepsilon(x) = \frac{1}{\varepsilon} f(\frac{x}{\varepsilon})$. Observe that for small $\varepsilon > 0$, the function f_ε is supported in $[0,1]$. Taking $\varepsilon = 2^{-n}$ for $n \in \mathbb{N}$

big enough we have that $\varepsilon x = \varepsilon \cdot x$ for all $x \in \mathbb{R}_+$. Thus we obtain $\pi_{-1}(x \cdot (\varepsilon y)) = \pi_{-1}(\varepsilon x) \cdot y)$, which implies that $\psi_{\varepsilon y}(x) = \psi_y(\varepsilon \cdot x)$. Summarizing, we have

$$\mathcal{F}_{\mathbb{F}} f_\varepsilon(y) = \int_0^\infty f_\varepsilon(x)\psi_y(x)dx = \int_0^\infty f(x)\psi_{\varepsilon y}(x)dx = \mathcal{F}_{\mathbb{F}} f(\varepsilon y).$$

Finally we obtain

$$\|\mathcal{F}_{\mathbb{F}} f\|_{p'} = \varepsilon^{1/p'}\|\mathcal{F}_{\mathbb{F}} f_\varepsilon\|_{p'} \le C\varepsilon^{1/p'}\|f_\varepsilon\|_p = C\|f\|_p.$$

Now a density argument shows that this holds for an arbitrary function $f \in L_p^X(\mathbb{R}_+)$.

To prove the implication $(iii) \Rightarrow (i)$ let us assume that X has Fourier type p with respect to the group $\mathbb{F}$. We claim that

$$\|\widehat{f}\|_{p'} \le C\|f\|_p \quad \text{for all} \quad f \in L_p^X(\mathbb{D}).$$

By assumption there is a constant $C > 0$ such that

$$\|\mathcal{F}_{\mathbb{F}} g\|_{L_{p'}^X(\mathbb{R}_+)} \le C\|g\|_p \quad \text{for all} \quad g \in L_p^X(\mathbb{R}_+). \tag{3.8}$$

By density it suffices to work with polynomials $f(t) = \sum_{k=0}^n x_k\psi_k(t)$, where $x_k \in X$. Let us now consider the function $h(s) = \sum_{k=0}^n x_k\chi_{[k,k+1]}(s)$. Then $h \in L_{p'}^X(\mathbb{R}_+)$. Consequently, we obtain

$$\left(\sum_{k=0}^n \|\widehat{f}(k)\|_X^{p'}\right)^{1/p'} = \left(\sum_{k=0}^n \|x_k\|_X^{p'}\right)^{1/p'} = \|h\|_{L_{p'}^X(\mathbb{R}_+)}.$$

We apply (3.8) to the function $g = \mathcal{F}_{\mathbb{F}}^{-1}h = \mathcal{F}_{\mathbb{F}} h \in L_p(\mathbb{R}_+)$. We get

$$\mathcal{F}_{\mathbb{F}} h(t) = \int_0^\infty \sum_{k=0}^n x_k\chi_{[k,k+1]}(s)\psi_t(s)ds = \sum_{k=0}^n x_k \int_k^{k+1} \psi_t(s)ds.$$

From (3.6) we get for $s \in [k, k+1]$

$$\begin{aligned}\int_k^{k+1} \psi_t(s)ds &= \int_k^{k+1} \psi_{[t]}(s)\psi_{[s]}(t)ds \\ &= \psi_k(t)\int_k^{k+1} \psi_{[t]}(s)ds = \begin{cases}\psi_k(t) & \text{for } 0 \le t < 1 \\ 0 & \text{for } t \ge 1\end{cases}.\end{aligned}$$

Hence we obtain

$$\|\mathcal{F}_{\mathbb{F}}h\|_{L_p^X(\mathbb{R}_+)} = \left(\int_0^1 \Big\|\sum_{k=0}^n x_k\psi_k(t)\Big\|_X^p dt\right)^{1/p} = \|f\|_{L_p^X([0,1])}.$$

Finally, it follows from (3.8) that

$$\left(\sum_{k=0}^n \|\widehat{f}(k)\|_X^{p'}\right)^{1/p'} = \|\mathcal{F}_{\mathbb{F}}g\|_{L_{p'}^X(\mathbb{R}_+)} \leq C\|g\|_p = C\|f\|_{L_p^X([0,1])}.$$

This finishes the proof. □

REMARKS

We restricted our attention only to the case of Banach spaces. Of course this result may be easily extended to the operator case:

THEOREM 3.2. *Let $1 < p < 2$. For an operator $T \in \mathcal{L}(X,Y)$ the following statements are equivalent*

(i) T has Fourier type p with respect to group $\mathbb{D}$.

(ii) T has Fourier type p with respect to group $\mathbb{D}^m$ for all $m \in \mathbb{N}$.

(iii) T has Fourier type p with respect to group $\mathbb{F}$.

(iv) T has Fourier type p with respect to group $\mathbb{F}^m$ for all $m \in \mathbb{N}$.

Moreover, in this case all norms coincide.

The Walsh system is a special case of a more general class of orthogonal functions systems, the so-called *Vilenkin system*. We refer the reader to the next chapter where we distinguish and investigate a certain class of operators between Banach spaces with help of Vilenkin systems.

Chapter 4

Bourgain's Hausdorff-Young inequality for cyclic groups

The starting point of our considerations in this chapter is a celebrated result of J. Bourgain from [Bou88] which says that B-convex spaces are just the spaces which have non-trivial Fourier type p (i.e. $p > 1$) with respect to the classical groups or the Cantor group. In the same paper Bourgain claimed that *the argument used in the proof of this result was easy adaptable to groups $\mathbb{Z}_m^\infty$ with $m > 2$.* Since all proofs of Bourgain's Theorem in the literature, e.g. in [GKKT98, PW98], only contain the original theorem for classical groups and for the Cantor group, our aim in this section is to give a complete proof of the above statement for infinite products of cyclic groups of prime power order. This chapter is organized as follows: The first section contains a brief summary of Vilenkin systems. In Section 4.2 we will be concerned with operators of Vilenkin type. The last Section provides a detailed proof of the above stated theorem for the cyclic groups of prime power order. This condition is unfortunatelly essential to the proof.

4.1 Vilenkin system

In this section we review some of the standard definitions and facts concerning Vilenkin systems. In the sequel, we shall denote the *m-adic intervals* in $[0,1]$ by

$$\Delta_n^{(k)} := \left[\frac{k}{m^n}, \frac{k+1}{m^n}\right) \qquad \text{for } \ 0 \leq k \leq m^n - 1.$$

For a fixed integer $m \geq 2$ every nonnegative integer $k \in \mathbb{N}_0$ has the unique *m-adic expansion*

$$k = \sum_{j=0}^{\infty} b_j m^j, \qquad \text{with } \ b_j \in \{0, 1, \ldots, m-1\}.$$

Observe that the sum is actually a finite sum. We shall now define a class of function systems by means of a generalization of Walsh systems. Let us extend the function

$$s_0(t) = \exp\left(\frac{2\pi i k}{m}\right) \quad \text{for every } \ t \in \Delta_1^{(k)}$$

to the whole line by periodicity of period one. The functions s_j given by

$$s_j(t) := s_0(m^j t) \quad \text{for every } \ t \in [0,1) \text{ and } j \in \mathbb{N}_0$$

are called *Rademacher functions in base m*. The *Vilenkin functions* v_k are given by

$$v_k(t) := \prod_{j=0}^{\infty} \big(s_j(t)\big)^{b_j}.$$

By the *Vilenkin system*, we mean $\mathcal{V} = \mathcal{V}^{(m)} := \{v_0, v_1, \ldots\}$. We will consider also *finite Vilenkin system* given by $\mathcal{V}_{m^n} := \{v_0, \ldots, v_{m^n-1}\}$. The above defined system was introduced by N. Ya. Vilenkin [Vil47]. It is easy to check that a Vilenkin system forms a complete orthonormal system in $L_2[0,1)$, i.e.

$$\int_0^1 v_i(t)\overline{v_j(t)}\, dt = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

and that $\{v_0, v_1, \ldots\}$ is a basis of $L_2[0,1)$. The principal significance of the system $\mathcal{V}^{(m)}$ is that it can be identified with the character group of $\mathbb{Z}_m^\infty$. Analogously $\mathcal{V}_{m^n}$ is the character group of $\mathbb{Z}_m^n$. Observe that we can completely determine the finite Vilenkin system $\mathcal{V}_{m^n}$ using the following *Vilenkin matrix* defined by

$$V_m := m^{-1/2} \exp\left(2\pi i \frac{hk}{m}\right) \quad \text{for } \quad h, k = 0, \ldots m^n - 1.$$

Furthermore, using tensor product notation we put

$$V_{m^n} := V_m \otimes V_m \otimes \cdots \otimes V_m.$$

For each $0 \leq h, k \leq m^n - 1$, denote by $v_h(\Delta_n^{(k)})$ the constant value which the Vilenkin function $v_h(t)$ takes on the m-adic interval $\Delta_n^{(k)}$. Notice that there exists a permutation π_n of $\{0, \ldots, m^n - 1\}$ such that the entries of the Vilenkin matrix V_{m^n} are given by

$$v_{hk}^{(m^n)} = m^{-n/2} v_{\pi_n(h)}(\Delta_n^{(k)}). \tag{4.1}$$

It turns out that the matrix $V_{m^n} := \left(v_{hk}^{(m^n)}\right)$ is unitary.

4.2 Operators of Vilenkin type

We begin this section with definition of Vilenkin type and cotype ideal norms. Let us fix a natural number $m \geq 2$ and assume that $1 \leq p, q < \infty$. We now introduce the notion of *Riemann ideal norms*, following the monograph [PW98, Ch. 3.3]. Let $\mathcal{A}_n = (a_0, \ldots, a_{n-1})$ and $\mathcal{B}_n = (b_0, \ldots, b_{n-1})$ be orthonormal systems in $L_2(M, \mu) \cap L_p(M, \mu)$ and $L_2(N, \nu) \cap L_q(N, \nu)$, respectively.
For $T \in \mathcal{L}(X, Y)$, we denote by $\varrho_p^{(q)}(T|\mathcal{B}_n, \mathcal{A}_n)$ the smallest constant $C \geq 0$ such that

$$\left(\int_N \Big\| \sum_{k=0}^{n-1} T x_k b_k(t) \Big\|^q d\nu(t) \right)^{1/q} \leq C \left(\int_M \Big\| \sum_{k=0}^{n-1} x_k a_k(t) \Big\|^p d\mu(t) \right)^{1/p} \tag{4.2}$$

holds for all $x_0 \ldots x_{n-1} \in X$. We refer to

$$\varrho_p^{(q)}(\mathcal{B}_n, \mathcal{A}_n) : T \longrightarrow \varrho_p^{(q)}(T|\mathcal{B}_n, \mathcal{A}_n)$$

as a *Riemann ideal norm*.
Let us denote by $\mathcal{I}_n$ the unit vector system in ℓ_2^n. By applying the above definition to the ortonormal systems $\mathcal{B}_n = \mathcal{V}_n$ and $\mathcal{A}_n = \mathcal{I}_n$ we obtain the following *Vilenkin ideal norms*. For $T \in \mathcal{L}(X, Y)$, we denote by $\varrho_p^{(q)}(T|\mathcal{V}_n, \mathcal{I}_n)$ the smallest constant $C \geq 0$ such that

$$\left(\int_0^1 \Big\| \sum_{k=0}^{n-1} T x_k v_k(t) \Big\|^q dt \right)^{1/q} \leq C \left(\sum_{k=0}^{n-1} \|x_k\|^p \right)^{1/p} \tag{4.3}$$

holds for all $x_0 \dots x_{n-1} \in X$. For simplicity of notation the inequality (4.3) will be written shortly as

$$\|(Tx_k)|\mathcal{V}_n\|_q \leq C\ \|(x_k)|\ell_p^n\|.$$

We refer to

$$\varrho_p^{(q)}(\mathcal{V}_n, \mathcal{I}_n) : T \longrightarrow \varrho_p^{(q)}(T|\mathcal{V}_n, \mathcal{I}_n)$$

as a *Vilenkin type ideal norm.*

We also consider the following counterpart of the above definition. For $T \in \mathcal{L}(X, Y)$, we denote by $\varrho_p^{(q)}(T|\mathcal{I}_n, \mathcal{V}_n)$ the smallest constant $C \geq 0$ such that

$$\left(\sum_{k=0}^{n-1} \|Tx_k\|^q\right)^{1/q} \leq\ C\left(\int_0^1 \Big\|\sum_{k=0}^{n-1} x_k v_k(t)\Big\|^p dt\right)^{1/p} \tag{4.4}$$

holds for all $x_0 \dots x_{n-1} \in X$. The inequality (4.4) can be written shortly as follows:

$$\|(Tx_k)|\ell_q^n\| \leq C\ \|(x_k)|\mathcal{V}_n\|_p.$$

We refer to

$$\varrho_p^{(q)}(\mathcal{I}_n, \mathcal{V}_n) : T \longrightarrow \varrho_p^{(q)}(T|\mathcal{I}_n, \mathcal{V}_n)$$

as a *Vilenkin cotype ideal norm.* In the case $p = q = 2$ we will omit the sub- and superscript.

Let $1 < p \leq 2$. An operator T is said to have *Vilenkin type p* if

$$\|T|\mathcal{VT}_p\| := \sup_n \varrho_p^{(p')}(T|\mathcal{V}_n, \mathcal{I}_n)$$

is finite. These operators form the Banach operator ideal

$$\mathcal{VT}_p := \mathcal{L}[\varrho_p^{(p')}(\mathcal{V}_n, \mathcal{I}_n)].$$

Now let $1 < p < 2$. An operator T is said to have *weak Vilenkin type p* if

$$\|T|\mathcal{VT}_p^{\mathrm{w}}\| := \sup_n n^{1/2-1/p} \varrho(T|\mathcal{V}_n, \mathcal{I}_n)$$

is finite. These operators form the Banach operator ideal

$$\mathcal{VT}_p^{\mathrm{w}} := \mathcal{L}[n^{1/2-1/p} \varrho(\mathcal{V}_n, \mathcal{I}_n)].$$

Observe that $\mathcal{VT}_1 = \mathcal{L}$. Furthermore, we define in a similar manner the Banach ideal of operators of *Vilenkin cotype* q denoted by $\mathcal{VC}_q$ and the Banach ideal of operators of *weak Vilenkin cotype* q denoted by $\mathcal{VC}_q^{\mathrm{w}}$. However, it turns out that

$$\mathcal{VC}_q = \mathcal{VT}_{q'} \quad \text{and} \quad \mathcal{VC}_q^{\mathrm{w}} = \mathcal{VT}_{q'}^{\mathrm{w}}.$$

This can be easily obtained from the following proposition.

PROPOSITION 4.1. *For every* $n \in \mathbb{N}$ *we have*

$$\varrho_p^{(q)}(\mathcal{I}_{m^n}, \mathcal{V}_{m^n}) = m^{n(1/p+1/q-1)}\, \varrho_p^{(q)}(\mathcal{V}_{m^n}, \mathcal{I}_{m^n}).$$

In particular we get

$$\varrho_p^{(q)\prime}(\mathcal{V}_{m^n}, \mathcal{I}_{m^n}) = m^{n(1/p+1/q-1)}\, \varrho_{p'}^{(q')}(\mathcal{V}_{m^n}, \mathcal{I}_{m^n}).$$

Proof. Observe that the ideal norms under consideration are stable with respect to permutations, i.e. $\varrho_p^{(q)}(\pi\circ\mathcal{A}_n, \pi\circ\mathcal{B}_n) = \varrho_p^{(q)}(\mathcal{A}_n, \mathcal{B}_n)$. Thus we assume by rearranging columns of the matrix V_{m^n} that the permutation π_n occuring in (4.1) is the identical permutation. It follows from (4.1) that

$$\int_0^1 \Big\| \sum_{k=0}^{m^n-1} x_k v_k(t) \Big\|^r dt = \frac{1}{m^n} \sum_{h=0}^{m^n-1} \Big\| \sum_{k=0}^{m^n-1} x_k m^{n/2} v_{hk}^{(m^n)} \Big\|^r.$$

Applying the above identity and taking

$$\|(x_k)|\mathcal{V}_{m^n}\|_r^r := \int_0^1 \Big\| \sum_{k=0}^{m^n-1} x_k v_k(t) \Big\|^r dt$$

and

$$\|(x_k)|V_{m^n}\|_r^r := \sum_{h=0}^{m^n-1} \Big\| \sum_{k=0}^{m^n-1} x_k v_{hk}^{(m^n)} \Big\|^r$$

yields

$$\|(x_k)|\mathcal{V}_{m^n}\|_r = m^{n(1/2-1/r)} \|(x_k)|V_{m^n}\|_r. \tag{4.5}$$

Moreover, we obtain that the inequalities

$$\|(Tx_k)|\ell_q^{m^n}\| \le C\|(x_k)|V_{m^n}\|_p \quad \text{and} \quad \|(T\widetilde{x_h})|V_{m^n}\|_q \le C\|(\widetilde{x_h})|\ell_p^{m^n}\| \tag{4.6}$$

are equivalent for arbitrary sequences (x_k) and $(\widetilde{x_h})$ given by

$$\widetilde{x_h} = \sum_{k=0}^{m^n-1} v_{hk}^{(m^n)} x_k \quad \text{and} \quad x_k = \sum_{h=0}^{m^n-1} v_{kh}^{(m^n)} \widetilde{x_h}.$$

Combining (4.5) with (4.6) we deduce that

$$\|(Tx_k)|\ell_q^{m^n}\| \leq m^{n(1/p-1/2)} C \|(x_k)|\mathcal{V}_{m^n}\|_p$$

and

$$\|(T\widetilde{x_h})|\mathcal{V}_{m^n}\|_q \leq m^{n(1/2-1/q)} C \|(\widetilde{x_k})|\ell_p^{m^n}\|$$

are also equivalent. Consequently, we have

$$m^{n(1/2+1/p)} \varrho_p^{(q)}(\mathcal{I}_{m^n}, \mathcal{V}_{m^n}) = \inf C = m^{n(1/q+1/2)} \varrho_p^{(q)}(\mathcal{V}_{m^n}, \mathcal{I}_{m^n}),$$

what gives the claim. $\square$

It follows that the ideal $\mathcal{VT}_p$ of operators of Vilenkin type p defined here is just the ideal of Fourier type p operators with respect to the compact abelian group $\mathbb{Z}_m^\infty$ as defined in Section 2.3 with equal norms.

In the sequel, we will consider also the following property. An operator T is said to have *Vilenkin subtype* if

$$\varrho(T|\mathcal{V}_n, \mathcal{I}_n) = o(\sqrt{n}).$$

These operators form the Banach operator ideal

$$\mathcal{VT} := \mathcal{L}_0[n^{-1/2} \varrho(\mathcal{V}_n, \mathcal{I}_n)].$$

We define the *Rademacher system* as $\mathcal{R} := \{r_1 \ldots, r_n\}$. Here r_k denotes the k-th Rademacher function. Analogously, an operator T is said to have *Rademacher subtype* if

$$\varrho(T|\mathcal{R}_n, \mathcal{I}_n) = o(\sqrt{n}).$$

These operators form the Banach operator ideal

$$\mathcal{RT} := \mathcal{L}_0[n^{-1/2} \varrho(\mathcal{R}_n, \mathcal{I}_n)].$$

We conclude this section with the following useful observation about Vilenkin ideal norms in the case $p = q = 2$.

PROPOSITION 4.2. *The sequence of Vilenkin ideal norms $(\varrho(\mathcal{V}_{m^n}, \mathcal{I}_{m^n}))$ is submultiplicative, i.e.*

$$\varrho(\mathcal{V}_{m^{n+k}}, \mathcal{I}_{m^{n+k}}) \leq \varrho(\mathcal{V}_{m^n}, \mathcal{I}_{m^n}) \circ \varrho(\mathcal{V}_{m^k}, \mathcal{I}_{m^k})$$

4.3 Main results

In the sequel, let m be a power of a prime. The following theorem summarizes the relations between type, weak type and subtype Vilenkin ideals.

THEOREM 4.3. *The ideals $\mathcal{VT}_p$ are linearly ordered with respect to p. More precisely, for $1 < p < q < 2$ we have*

$$\mathcal{VT}_2 \subset \mathcal{VT}_q \subset \mathcal{VT}_q^{\mathrm{w}} \subset \mathcal{VT}_p \subset \mathcal{VT}. \tag{4.7}$$

In particular we have

$$\bigcup_{1<p\leq 2} \mathcal{VT}_p = \bigcup_{1<p<2} \mathcal{VT}_p^{\mathrm{w}}. \tag{4.8}$$

The proof of the inclusion $\mathcal{VT}_p \subset \mathcal{VT}$ is obvious. Applying Hölder's inequality shows the inclusion $\mathcal{VT}_q \subset \mathcal{VT}_q^{\mathrm{w}}$. For the remaining non-trivial inclusion we need to prove the following lemma.

LEMMA 4.4. *Let $q \geq 2$. For any finite subset $\mathbb{F}$ of $\mathbb{N}$ such that card$(\mathbb{F}) = m^n$ the following inequality holds*

$$\varrho_2^{(q)}(\mathcal{V}(\mathbb{F}), \mathcal{I}(\mathbb{F})) \leq 2\varrho_2^{(q)}(\mathcal{V}_{m^n}, \mathcal{I}_{m^n}).$$

Proof. Let $\mathcal{S}_N = (\gamma_1, \ldots, \gamma_N)$ be any system of characters on the compact abelian group $G = \mathbb{Z}_m^\infty$. Consider the function f_k given by

$$f_k(c,t) := \prod_{h=1}^{n} \gamma_k(t_h)^{c_h} = \gamma_k\Big(\sum_{h=1}^{n} c_h t_h\Big) \tag{4.9}$$

for $t = (t_1, \ldots, t_n) \in G^n$ and $c = (c_1, \ldots, c_n) \in \mathbb{Z}_m^n$. Let us fix $t_1, \ldots, t_n \in G$ and develop the function f_k in the Vilenkin series

$$f_k(c,t) = \sum_{i<m^n} s_{k,i}(t) v_i(c), \tag{4.10}$$

where the coefficients are given by

$$s_{k,i}(t) = \frac{1}{m^n} \sum_{c \in \mathbb{Z}_m^n} \gamma_k\Big(\sum_{h=1}^{n} c_h t_h\Big)\overline{v_i(c)}.$$

An easy computation shows that the functions $f_k(\cdot, t)$ are characters on $\mathbb{Z}_m^n$. Indeed, we have

$$\begin{aligned} f_k(c \oplus d, t) &= \gamma_k\Big(\sum_{h=1}^{n}(c_h \oplus d_h)t_h\Big) = \gamma_k\Big(\sum_{h=1}^{n} c_h t_h \oplus \sum_{h=1}^{n} d_h t_h\Big) \\ &= \gamma_k\Big(\sum_{h=1}^{n} c_h t_h\Big) \cdot \gamma_k\Big(\sum_{h=1}^{n} d_h t_h\Big) = f_k(c,t) \cdot f_k(d,t). \end{aligned}$$

Observe that $s_{k,i}(t) = 1$ only for $v_i = f_k$ and $s_{k,i}(t) = 0$ in other cases. Consequently, we have that $s_{k,i}(t) \geq 0$. Let $x_1, \dots x_N \in X$. For $r \in G$ define

$$x_i(r,t) := \sum_{k=1}^{N} x_k s_{k,i}(t)\gamma_k(r). \tag{4.11}$$

It follows easily that

$$\|x_i(r,t)\| \leq \sum_{k=1}^{N} \|x_k\| s_{k,i}(t). \tag{4.12}$$

For fixed $c = (c_1, \dots, c_n) \in \mathbb{Z}_m^n$ applying (4.9), (4.10) and (4.11) we obtain

$$\begin{aligned} \sum_{k=1}^{N} Tx_k \left[\gamma_k\Big(\sum_{h=1}^{n} c_h t_h\Big)\right] \gamma_k(r) &= \sum_{k=1}^{N} Tx_k \sum_{i \leq m^n} s_{k,i}(t) v_i(c)\gamma_k(r) \\ &= \sum_{i < m^n} Tx_i(r,t) v_i(c). \end{aligned}$$

Using the invariance of the Haar measure we obtain for $c = (c_1, \dots, c_n) \in \mathbb{Z}_m^n$ and

$t_1, \ldots, t_n \in G$ that

$$\begin{aligned}
\|(Tx_k)|\mathcal{S}_N\|_q &= \left(\int_G \Big\| \sum_{k=1}^N Tx_k \gamma_k(r) \Big\|^q d\mu(r)\right)^{1/q} \\
&= \left(\int_G \Big\| \sum_{k=1}^N Tx_k \Big[\gamma_k\Big(\sum_{h=1}^n c_h t_h\Big)\Big]\gamma_k(r) \Big\|^q d\mu(r)\right)^{1/q} \\
&= \left(\int_G \Big\| \sum_{i<m^n} Tx_i(r,t) v_i(c) \Big\|^q d\mu(r)\right)^{1/q} \qquad (4.13)
\end{aligned}$$

Averaging over $c = (c_1, \ldots, c_n) \in \mathbb{Z}_m^n$ then gives

$$\|(Tx_k)|\mathcal{S}_N\|_q = \left(\frac{1}{m^n} \sum_{c \in \mathbb{Z}_m^n} \int_G \Big\| \sum_{i<m^n} Tx_i(r,t) v_i(c) \Big\|^q d\mu(r)\right)^{1/q}.$$

Combining type property with inequality (4.12) yields

$$\begin{aligned}
&\Big(\frac{1}{m^n} \sum_{c \in \mathbb{Z}_m^n} \Big\| \sum_{i<m^n} Tx_i(r,t) v_i(c) \Big\|^q \Big)^{1/q} \leq \\
&\leq \varrho_2^{(q)}(T|\mathcal{V}_{m^n}, \mathcal{I}_{m^n}) \left(\sum_{i<m^n} \|x_i(r,t)\|^2\right)^{1/2} \\
&\leq \varrho_2^{(q)}(T|\mathcal{V}_{m^n}, \mathcal{I}_{m^n}) \left(\sum_{i<m^n} \left[\sum_{k=1}^N \|x_k\| s_{k,i}(t)\right]^2\right)^{1/2}.
\end{aligned}$$

Since the right hand side of the above inequality does not depend on $r \in G$ we obtain by integration

$$\begin{aligned}
&\left(\frac{1}{m^n} \sum_{c \in \mathbb{Z}_m^n} \int_G \Big\| \sum_{i<m^n} Tx_i(r,t) v_i(c) \Big\|^q d\mu(r)\right)^{1/q} \leq \\
&\leq \varrho_2^{(q)}(T|\mathcal{V}_{m^n}, \mathcal{I}_{m^n}) \left(\sum_{i<m^n} \left|\sum_{k=1}^N \|x_k\| s_{k,i}(t)\right|^2\right)^{1/2}.
\end{aligned}$$

Consequently, equation (4.13) yields

$$\|(Tx_k)|\mathcal{S}_N\|_q \leq \varrho_2^{(q)}(T|\mathcal{V}_{m^n}, \mathcal{I}_{m^n}) \left(\sum_{i<m^n} \left| \sum_{k=1}^{N} \|x_k\| s_{k,i}(t) \right|^2 \right)^{1/2}.$$

Since the left-hand side of the above inequality does not depend on the elements $t_1, \ldots, t_n \in G$, integration yields

$$\begin{aligned} &\|(Tx_k)|\mathcal{S}_N\|_q \leq \\ &\quad \leq \varrho_2^{(q)}(T|\mathcal{V}_{m^n}, \mathcal{I}_{m^n}) \left(\sum_{i<m^n} \int_{G^n} \left| \sum_{k=1}^{N} \|x_k\| s_{k,i}(t) \right|^2 d\mu^{(n)}(t) \right)^{1/2}. \end{aligned} \tag{4.14}$$

Furthermore we obtain

$$\begin{aligned} &\sum_{i<m^n} \int_{G^n} \left| \sum_{k=1}^{N} \|x_k\| s_{k,i}(t) \right|^2 d\mu^{(n)}(t) = \\ =&\frac{1}{m^{2n}} \sum_{i<m^n} \int_{G^n} \left| \sum_{k=1}^{N} \|x_k\| \sum_{c\in\mathbb{Z}_m^n} f_k(c,t)\overline{v_i(c)} \right|^2 d\mu^{(n)}(t) \\ =&\frac{1}{m^{2n}} \sum_{i<m^n} \int_{G^n} \left[\sum_{k,l=1}^{N} \|x_k\|\|x_l\| \sum_{c,d\in\mathbb{Z}_m^n} f_k(c,t)\overline{f_l(d,t)v_i(c)}v_i(d) \right] d\mu^{(n)}(t) \\ =&\frac{1}{m^n} \sum_{k,l=1}^{N} \|x_k\|\|x_l\| \sum_{c,d\in\mathbb{Z}_m^n} \int_{G^n} f_k(c,t)\overline{f_l(d,t)} d\mu^{(n)}(t) \frac{1}{m^n} \sum_{i<m^n} \overline{v_i(c)}v_i(d). \end{aligned}$$

Hence taking into account that

$$\frac{1}{m^n} \sum_{i<m^n} \overline{v_i(c)}v_i(d) = \begin{cases} 1 & \text{for } c = d, \\ 0 & \text{otherwise.} \end{cases}$$

yields together with the fact that the $f(c,\cdot)$ are characters on G^n

$$\begin{aligned}&\sum_{i<m^n}\int_{G^n}\Big|\sum_{k=1}^{N}\|x_k\|s_{k,i}(t)\Big|^2 d\mu^{(n)}(t) = \\ &\quad = \frac{1}{m^n}\Big(\sum_{k,l=1}^{N}\|x_k\|\|x_l\|\Big)\operatorname{card}\left(\left\{c\in\mathbb{Z}_m^n : \int_{G^n} f_k(c,t)\overline{f_l(c,t)}d\mu^{(n)}(t)=1\right\}\right).\end{aligned} \tag{4.15}$$

Now the assumption, that m is a prime power implies that

$$\begin{aligned}\int_{G^n} f_k(c,t)\overline{f_l(c,t)}d\mu^{(n)}(t) &= \int_{G^n}\prod_{h=1}^{n}\gamma_k(t_h)^{c_h}\prod_{h=1}^{n}\gamma_l(t_h)^{-c_h}d\mu^{(n)}(t)\\ &= \prod_{h=1}^{n}\int_G \gamma_k(s)^{c_h}\gamma_l(s)^{-c_h}ds = \begin{cases}1 & \text{for } c=(0,\dots,0) \text{ or } k=l,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Now (4.15) gives

$$\begin{aligned}\sum_{i<m^n}\int_{G^n}&\left|\sum_{k=1}^{N}\|x_k\|s_{k,i}(t)\right|^2 d\mu^{(n)}(t) = \\ &=\frac{1}{m^n}\left(\Big(\sum_{k=1}^{N}\|x_k\|\Big)^2+(m^n-1)\sum_{k=1}^{N}\|x_k\|^2\right).\\ &\le\frac{1}{m^n}(N+m^n-1)\sum_{k=1}^{N}\|x_k\|^2.\end{aligned} \tag{4.16}$$

Finally, combining (4.14) and (4.16) we arrive at

$$\varrho_2^{(q)}(\mathcal{S}_N,\mathcal{I}_N) \le \left(\frac{N+m^n-1}{m^n}\right)^{1/2}\cdot\varrho_2^{(q)}(\mathcal{V}_{m^n},\mathcal{I}_{m^n}) \quad \text{for} \quad N\le m^n.$$

Taking for $\mathcal{S}_N$ any Vilenkin system $\mathcal{V}(\mathbb{F})$ with $\operatorname{card}(\mathbb{F})=m^n$ finishes the proof. $\square$

The preceding Lemma is the key ingredient in the proof of the following fact.

PROPOSITION 4.5. *Assume that $1<p<2$ and $q\ge 2$. Then the following statements are equivalent:*

(i) There exists a constant $A \geq 1$ such that

$$\varrho_2^{(q)}(T|\mathcal{V}_n, \mathcal{I}_n) \leq A\, n^{1/p-1/2} \quad \textit{for} \quad n \in \mathbb{N}.$$

(ii) There exists a constant $B \geq 1$ such that

$$\varrho_{p,1}^{(q)}(T|\mathcal{V}_n, \mathcal{I}_n) \leq B \quad \textit{for} \quad n \in \mathbb{N}.$$

Proof. The proof is based on estimates of Lorentz norms collected in Lemma 2.1. The inequality

$$\|(x_k)|\ell_{p,1}^n\| \leq C n^{1/p-1/2} \|(x_k)|\ell_2^n\|$$

with constant $C = 4p/(p-2)$ shows the implication $(ii) \Rightarrow (i)$. For the reverse implication let us fix $x_0, \ldots, x_{n-1} \in X$ and consider a partition of $\{0, \ldots, n-1\}$ into pairwise disjoint subsets $\mathbb{F}_i$ such that

$$\sum_{i=0}^{n-1} |\mathbb{F}_i|^{1/p-1/2} \Big(\sum_{k \in \mathbb{F}_i} \|x_k\|^2 \Big)^{1/2} \leq 2\|(x_k)|\ell_{p,1}^n\|.$$

Applying Lemma 4.4 we conclude that

$$\left(\int_0^1 \Big\| \sum_{k=0}^{n-1} T x_k v_k(t) \Big\|^q dt \right)^{1/q} \leq \sum_{i=0}^{n-1} \left(\int_0^1 \Big\| \sum_{k \in \mathbb{F}_i} T x_k v_k(t) \Big\|^q dt \right)^{1/q}$$

$$\begin{aligned}
&\leq \sum_{i=0}^{n-1} 2 \varrho_2^{(q)}(\mathcal{V}(\mathbb{F}_i), \mathcal{I}(\mathbb{F}_i)) \Big(\sum_{k \in \mathbb{F}_i} \|x_k\|^2 \Big)^{1/2} \\
&\leq 2A \sum_{i=0}^{n-1} |\mathbb{F}_i|^{1/p-1/2} \Big(\sum_{k \in \mathbb{F}_i} \|x_k\|^2 \Big)^{1/2} \leq 2A \|(x_k)|\ell_{p,1}^n\|.
\end{aligned}$$

Consequently, $\varrho_{p,1}^{(q)}(\mathcal{V}_n, \mathcal{I}_n) \leq B := 2A$ which finishes the proof. □

The interpretation of the last proposition in terms of operator ideals reads as follows

PROPOSITION 4.6. *Let $1 < p < 2$. Then we have*

$$\mathcal{VT}_p^{\mathrm{w}} = \mathcal{L}[\varrho_{p,1}^{(q)}(\mathcal{V}_n, \mathcal{I}_n)] = \mathcal{L}[n^{1/2-1/p} \varrho_2^{(q)}(\mathcal{V}_n, \mathcal{I}_n)].$$

Roughly speaking the above Proposition says that the Banach operator ideals $\mathcal{VT}_p^{\mathrm{w}}$ are independent of the superscript q.
In order to finish the proof of the remaining inclusion in Theorem 4.3 we combine the Proposition above with the following one:

PROPOSITION 4.7. *Let* $1 < p < r \leq 2$. *Then there exists a constant* $C > 1$ *depending only on* p *and* r *such that*

$$\varrho_p^{(p')}(\mathcal{V}_n, \mathcal{I}_n) \leq C \varrho_{r,1}^{(r')}(\mathcal{V}_n, \mathcal{I}_n). \tag{4.17}$$

In terms of operator ideals the above inequality reads as follows:

$$\mathcal{L}[\varrho_{r,1}^{(r')}(\mathcal{V}_n, \mathcal{I}_n)] \subseteq \mathcal{VT}_p.$$

Proof. The basic idea of the proof is to use interpolation theory. Define the operator

$$V_T : (x_i) \longrightarrow \sum_{i=0}^{n-1} T x_i v_i.$$

Then we have

$$\begin{aligned} \|V_T : \ell_{r,1}^n(X) \to L_{r'}^Y(G)\| &= \varrho_{r,1}^{(r')}(T|\mathcal{V}_n, \mathcal{I}_n), \\ \|V_T : \ell_1^n(X) \to L_\infty^Y(G)\| &= \|T\| \end{aligned}$$

and

$$\|V_T : \ell_p^n(X) \to L_{p'}^Y(G)\| = \varrho_p^{(p')}(T|\mathcal{V}_n, \mathcal{I}_n).$$

Let us now recall necessary interpolation formulas for the real interpolation method:

$$\big(\ell_{r,1}^n(X), \ell_1^n(X)\big)_{\theta,p} = \ell_p^n(X)$$

and

$$\big(L_{r'}^Y(G), L_\infty^Y(G)\big)_{\theta,p} = L_{p',p}^Y(G),$$

where θ is given by $1/p = (1-\theta)/r + \theta/1$. These equalities are meant as isomorphisms, not isometries. The best general reference here is [BL78].
Moreover, it is well-known that

$$L_{p',p}^Y(G) \subseteq L_{p'}^Y(G) \quad \text{for} \quad p \leq p'.$$

Hence

$$\varrho_p^{(p')}(T|\mathcal{V}_n,\mathcal{I}_n) \le C\varrho_{r,1}^{(r')}(T|\mathcal{V}_n,\mathcal{I}_n)^{1-\theta}\|T\|^{\theta}.$$

The last inequality together with $\|T\| \le \varrho_{r,1}^{(r')}(T|\mathcal{V}_n,\mathcal{I}_n)$ establishes (4.17) and hence completes the proof. □

Next we give an example which states that the inclusions $\mathcal{VT}_q \subset \mathcal{VT}_q^{\mathrm{w}} \subset \mathcal{VT}_p$ are strict. In order to find such examples we will use the diagonal operator $D_\lambda, \lambda > 0$ given by

$$D_\lambda : (\xi_k) \to (k^{-\lambda}\xi_k).$$

EXAMPLE. For $1 < p < q < 2$ we have

$$L_p, L_{p'} \in \mathsf{VT}_p \setminus \mathsf{VT}_q^{\mathrm{w}}$$

and

$$D_{1/p'} \in \mathcal{VT}_p^{\mathrm{w}}(\ell_1) \setminus \mathcal{VT}_p(\ell_1).$$

In order to examine the first example observe that $\varrho_p^{(p')}(L_p|\mathcal{V}_n,\mathcal{I}_n) = 1$ which implies that $L_p \in \mathsf{VT}_p$. It follows from Proposition 4.1 we also have $L_{p'} \in \mathsf{VT}_p$. Moreover, by [PW98, 3.7.11] we obtain

$$\varrho(L_p|\mathcal{V}_n,\mathcal{I}_n) \asymp \varrho(L_p'|\mathcal{V}_n,\mathcal{I}_n) \asymp n^{1/p-1/2},$$

which gives $L_p, L_{p'} \notin \mathsf{VT}_p^{\mathrm{w}}$. To deduce the second property observe that by [PW98, 3.7.15] we have

$$\varrho_p^{(p')}(D_{1/p'} : \ell_1 \to \ell_1|\mathcal{V}_n,\mathcal{I}_n) = \Big(\sum_{k=1}^{n}\frac{1}{k}\Big)^{1/p'} \asymp (1+\log n)^{1/p'}$$

and

$$\varrho_2^{(2)}(D_{1/p'} : \ell_1 \to \ell_1|\mathcal{V}_n,\mathcal{I}_n) = \Big(\sum_{k=1}^{n}k^{-2/p'}\Big)^{1/2} \asymp n^{1/p-1/2}.$$

To round off our discussion of the properties of Vilenkin type operators, we present a beautiful consequence of Theorem 4.3 in the setting of Banach spaces. The equality (4.8) reads then as follows

THEOREM 4.8.

$$\bigcup_{1<p\leq 2} \mathsf{VT}_p = \bigcup_{1<p<2} \mathsf{VT}_p^{\mathrm{w}} = \mathsf{VT}.$$

The last equality follows as in the Walsh case in [PW98, 6.3.10] from Proposition 4.2. Notice that the last equality does not occur in the operator case. To see this, consider the diagonal operator $C_\lambda, \lambda \geq 0$ on $\left(\bigoplus l_\infty^{m^k}\right)_2$ given by $C_\lambda(x_k) = (k^{-\lambda} x_k)$ with $x_k \in \ell_\infty^{m^k}$. Moreover, it can be computed analogously to [PW98, 6.3.11] that $\varrho(C_\lambda|\mathcal{V}_n, \mathcal{I}_n) \asymp n^{1/2}(1+\log n)^{-\lambda}$. Finally, the Vilenkin counterpart of Bourgain's Theorem, see [Bou88] reads as follows in the language of operator ideals.

THEOREM 4.9.

$$\mathcal{VT} = \mathcal{RT}.$$

Proof. The inclusion $\mathcal{VT} \subset \mathcal{RT}$ follows from the inequality $\varrho(\mathcal{R}_n, \mathcal{I}_n) \leq \frac{\pi}{2}\varrho(\mathcal{V}_n, \mathcal{I}_n)$, which is a consequence of the principle of contraction, see [PW98, 3.7.9]. The reverse inclusion can be shown by using Hinrichs's inequality, see [PW98, 4.11.18]. $\square$

It follows from Theorem 4.8 and 4.9 that Banach spaces of non-trivial Vilenkin type are just the B-convex spaces.

Chapter 5

Fourier type 2 operators with respect to locally compact abelian groups

The main result of this chapter states that among all ideals of Fourier type 2 operators with respect to G the ideal of Fourier type 2 operators with respect to classical groups is the smallest. The results comprising this chapter were obtained in the joint work [HP03] with A. Hinrichs.

5.1 Introduction

We start our considerations in this chapter by recalling Kwapień's celebrated isomorphic characterization of Hilbert spaces. Its main assertion is that a Banach space X with the property that X-valued functions satisfy a "Parseval inequality" is already isomorphic to a Hilbert space, see [Kwa72]. The informal term "Parseval inequality" here refers to the existence of a constant $c > 0$ such that either

$$\int_0^1 \Big\| \sum_{k=-N}^{N} e^{2\pi ikt} x_k \Big\|^2 dt \le c^2 \sum_{k=-N}^{N} \|x_k\|^2$$

or

$$\sum_{k=-N}^{N} \|x_k\|^2 \le c^2 \int_0^1 \Big\| \sum_{k=-N}^{N} e^{2\pi ikt} x_k \Big\|^2 dt$$

holds for all $N = 1, 2, \ldots$ and $x_{-N}, \ldots, x_N \in X$. In other words, we have a uniform estimate of the L_2-norm of an X-valued function in terms of the l_2-norm of its Fourier coefficients or vice versa.

It is not necessary to use trigonometric functions in this characterization, any complete system of characters on an infinite compact abelian group will do. A basic open question here is, whether a similar characterization can be derived for linear and bounded operators factoring through a Hilbert space. More precisely, is it true that any operator T between Banach spaces X and Y, for which there exists a constant $c > 0$ such that

$$\int_0^1 \Big\| \sum_{k=-N}^{N} T e^{2\pi i k t} x_k \Big\|^2 dt \leq c^2 \sum_{k=-N}^{N} \|x_k\|^2 \tag{5.1}$$

or

$$\sum_{k=-N}^{N} \|T x_k\|^2 \leq c^2 \int_0^1 \Big\| \sum_{k=-N}^{N} e^{2\pi i k t} x_k \Big\|^2 dt \tag{5.2}$$

holds for all $N = 1, 2, \ldots$ and $x_{-N}, \ldots, x_N \in X$, can be factored as $T = AB$ with linear bounded operators B from X to a Hilbert space H and A from H to Y? Let us denote by $\mathcal{H}$ the class of all operators T factoring through a Hilbert space, that is for which there are a Hilbert space H and linear bounded operators A and B as above such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "T = AB"] \arrow[dr, "B"'] & & Y \\ & H \arrow[ur, "A"'] & \end{tikzcd}$$

The above question which was posed e.g in [PW98] then asks whether every operator of Fourier type 2 with respect to an arbitrary lca group G factors through Hilbert space? In other words we ask whether $\mathcal{FT}_2^G = \mathcal{H}$ holds. By the ideal property and Kwapień's Theorem we obtain immediately that $\mathcal{H} \subset \mathcal{FT}_2^G$. An affirmative answer to this question would mean that the concept of Fourier type 2 does not depend on the underlying lca group G. Other interesting aspects of this problem were studied in [Hin01b].

The main purpose of this chapter is to show that among those classes of operators for which a Parseval inequality with respect to some compact abelian group G holds, the class where G is the torus group $\mathbb{T}$ or the group $\mathbb{Z}$ of integers is the smallest. This latter class consists just of the operators for which (5.1) (or, equivalently, (5.2)) holds. It will be convenient to cast this in the language of Fourier type with respect to locally compact abelian groups (see Section 2.3), even if we are only interested in the case of Fourier type 2 operators. In the case $p = 2$, we have coinciding norms:

$$\|T \,|\, \mathcal{FT}_2^{\mathbb{Z}^k}\| = \|T \,|\, \mathcal{FT}_2^{\mathbb{T}^m}\| = \|T \,|\, \mathcal{FT}_2^{\mathbb{R}^n}\| \tag{5.3}$$

for all $k, m, n = 1, 2, \ldots$, see [And98]. For definitions and more information on the concept of Fourier type p for operators and spaces, the reader is referred to Section 2.3. We now state the main result of this chapter.

THEOREM 5.1. *Any operator of Fourier type 2 with respect to the classical groups has Fourier type 2 with respect to all lca groups. More precisely,*

$$\mathcal{FT}_2^{\mathbb{T}} \subseteq \mathcal{FT}_2^{G} \quad \textit{and} \quad \|T \,|\, \mathcal{FT}_2^{G}\| \leq \|T \,|\, \mathcal{FT}_2^{\mathbb{T}}\|$$

holds for all lca groups G and all $T \in \mathcal{FT}_2^{\mathbb{T}}$.

Forerunners of this theorem were proved in [Hin01a] (for the case where G is the Cantor group) and in [CHL03] (for the case that G is a closed subgroup of $\mathbb{R}^n$). In the next section, we give a proof of the inequality in Theorem 5.1 for a finite group G. This will be the basis for the proof of the general case, which is accomplished invoking some known facts from the theory of Fourier type and the structure theorem for lca groups.

5.2 Finite groups and Vilenkin case

The purpose of this section is to establish the inequality

$$\|T \,|\, \mathcal{FT}_2^{G}\| \leq \|T \,|\, \mathcal{FT}_2^{\mathbb{T}}\| \tag{5.4}$$

in Theorem 5.1 for finite abelian groups G. Since any finite abelian group is a product of finite cyclic groups, Fubini's Theorem and (5.3) show that it is enough to prove (5.4) for the case of a cyclic group. We follow the simulation of characters approach used in [Hin01a]. So let $m = 2, 3, \ldots$ be fixed and let $\mathbb{Z}_m$ be the cyclic group of order m. The characters $\chi_0, \ldots, \chi_{m-1}$ are given by

$$\chi_h(k) = \exp(-2\pi i h k/m).$$

Then $\|T \,|\, \mathcal{FT}_2^{\mathbb{Z}_m}\|$ is the smallest constant c such that

$$\frac{1}{m} \sum_{k=0}^{m-1} \Big\| \sum_{h=0}^{m-1} T x_h e^{2\pi i h k/m} \Big\|^2 \le c^2 \sum_{k=0}^{m-1} \|x_k\|^2 \tag{5.5}$$

holds for all $x_0, \ldots, x_{m-1} \in X$. Let $\omega_k = \exp(2\pi i k/m)$ for $k = 0, \ldots, m-1$ be the m-th roots of unity. Obviously, inequality (5.5) is equivalent to the inequality

$$\int_0^1 \Big\| \sum_{h=0}^{m-1} T x_h \chi_h(t) \Big\|^2 dt \le c^2 \sum_{k=0}^{m-1} \|x_k\|^2, \tag{5.6}$$

where the functions χ_h are now defined on $[0,1]$ by

$$\chi_h(t) = \omega_k^h \text{ for } k = 0, \ldots, m-1 \text{ and } t \in [k/m, (k+1)/m).$$

The key to our proof of inequality (5.4) is the following simple lemma. As usual, the spectrum of a trigonometric polynomial $p(t) = \sum_k \lambda_k \exp(2\pi i k t)$ is the set $\operatorname{spec}(p) = \{k \in \mathbb{Z} \,:\, \lambda_k \neq 0\}$.

LEMMA 5.2. *For all $\varepsilon > 0$ and $m = 2, 3, \ldots$ there exist trigonometric polynomials $p_0, \ldots, p_{m-1}$ with mutually disjoint spectra and a set $A \subset [0,1]$ with measure $1 - \varepsilon$ such that $|\chi_h(t) - p_h(t)| < \varepsilon$ for $t \in A$ and $h = 0, 1, \ldots, m-1$. Moreover, we can choose these polynomials with $\|p_h\|_\infty \le 1$.*

Proof. Let $K_N(t) = \frac{1}{N+1} \frac{\sin^2 (N+1)\pi t}{\sin^2 \pi t}$ be the N-th Fejér kernel on $[0,1]$. Let

$$p_h^{(N)}(t) = \int_0^1 \chi_h(s) K_N(t-s)\, ds$$

be the N-th sum in the Fejér summation method of the Fourier series of χ_h. The properties of the Fejér kernel imply that $\|p_h^{(N)}\|_\infty \le \|\chi_h\|_\infty = 1$. An easy calculation shows that the Fourier coefficients of χ_h satisfy

$$\hat{\chi}_h(j) \neq 0 \text{ only if } j \equiv h \mod m.$$

Hence also $\operatorname{spec}(p_h^{(N)}) \subset \{j \in \mathbb{Z} : j \equiv h \mod m\}$ which implies that the spectra of $p_0^{(N)}, \dots, p_{m-1}^{(N)}$ are mutually disjoint for fixed N. Moreover, on any compact subset $A \subset [0,1]$ avoiding the points of discontinuity of the functions χ_h, $p_h^{(N)}$ converges uniformly to χ_h if $N \to \infty$. Choosing $p_h = p_h^{(N)}$ for N large enough then proves the lemma. $\square$

Proof of inequality (5.4) for finite G. As already pointed out we may and do assume that $G = \mathbb{Z}_m$ for some m. Let $x_0, \dots, x_n \in X$ and $\varepsilon > 0$. We can also assume without loss of generality that $\|T\| = 1$ and $\sum_{k=0}^n \|x_k\|^2 = 1$. Choose trigonometric polynomials $p_0, \dots, p_n$ and a set $A \subset [0,1]$ as in the previous lemma. Let us denote the spectrum of p_h by M_h and let p_h be given as $p_h(t) = \sum_{j \in M_h} \lambda_j e^{2\pi ijt}$. Then we find that

$$\Big(\int_0^1 \Big\|\sum_{h=0}^n \chi_h(t) T x_h\Big\|^2 dt\Big)^{1/2} \le$$
$$\underbrace{\Big(\int_0^1 \Big\|\sum_{h=0}^n (\chi_h(t) - p_h(t)) T x_h\Big\|^2 dt\Big)^{1/2}}_{:=I_1} + \underbrace{\Big(\int_0^1 \Big\|\sum_{h=0}^n p_h(t) T x_h\Big\|^2 dt\Big)^{1/2}}_{:=I_2}.$$

From (5.1) and the disjointness of the M_h we get

$$I_2 = \int_0^1 \Big\|\sum_{h=0}^n \sum_{j \in M_h} e^{2\pi ijt} T(\lambda_j x_h)\Big\|^2 dt \ \le\ \|T \,|\, \mathcal{FT}_2^{\mathbb{T}}\|^2 \sum_{h=0}^n \sum_{j \in M_h} |\lambda_j|^2 \|x_h\|^2.$$

Also $\|p_h\|_\infty \le 1$ implies that $\|p_h\|_2 \le 1$. Then Parseval's equality gives that $\sum_{j \in M_h} |\lambda_j|^2 \le 1$. Hence we find that $I_2 \le \|T \,|\, \mathcal{FT}_2^{\mathbb{T}}\|^2$.
Using triangle and Cauchy-Schwartz inequalities we estimate

$$I_1 \ \le\ \int_0^1 \Big(\sum_{h=0}^n |\chi_h(t) - p_h(t)| \|T x_h\|\Big)^2 dt \ \le\ \sum_{h=0}^n \int_0^1 |\chi_h(t) - p_h(t)|^2 dt.$$

The latter integrand is bounded by ε^2 on A and by 4 outside of A. For $\varepsilon \le 1$ we have $I_1 \le (n+1)(\varepsilon^2 + 4\varepsilon) \le 5\varepsilon(n+1)$. We obtain that

$$\Big(\int_0^1 \Big\|\sum_{h=0}^n \chi_h(t) T x_h\Big\|^2 dt\Big)^{1/2} \le \|T \,|\, \mathcal{FT}_2^{\mathbb{T}}\| + \sqrt{5\varepsilon(n+1)}.$$

Since $\varepsilon > 0$ was arbitrary, letting $\varepsilon \to 0$ implies inequality (5.4) as we wanted to show. $\square$

We conclude this section by showing a special case of our main result. Let us recall briefly some definitions. For a fixed integer $m \geq 2$ and every nonnegative integer $k \in \mathbb{N}_0$ with m-adic expansion $k = \sum_{j=0}^{\infty} b_j m^j$ with $b_j \in \{0, 1, \ldots, m-1\}$ the *Vilenkin functions* are given by

$$v_k(t) = \prod_{j=0}^{\infty} \big(s_j(t)\big)^{b_j},$$

where the functions s_j with $j \in \mathbb{N}_0$ are defined by

$$s_0(t) = \exp\left(\frac{2\pi i k}{m}\right) \quad \text{for every } \ t \in \left[\frac{k}{m}, \frac{k+1}{m}\right)$$

and

$$s_j(t) = s_0(m^j t) \quad \text{for every } \ t \in [0,1) \text{ and } j \in \mathbb{N}_0.$$

By the *Vilenkin system*, we mean $\mathcal{V} = \{v_0, v_1, \ldots\}$. An operator $T \in \mathcal{L}(X, Y)$ is said to be of Vilenkin type 2 (or equivalently of Fourier type 2 with respect to the group $\mathbb{Z}_m^{\infty}$) if there exists a constant $c > 0$ such that

$$\int_0^1 \Big\| \sum_{k=0}^{\infty} T x_k v_k(t) \Big\|^2 dt \ \leq \ c^2 \sum_{k=0}^{\infty} \|x_k\|^2. \tag{5.7}$$

Below we show how to construct polynomials as in Lemma 5.2 in this case. Observe that the proof of inequality (5.4) may be easily adapted to show that every operator of Fourier type 2 with respect to the classical groups has also Vilenkin type 2.

PROPOSITION 5.3. *For all $\delta > 0$ and $n = 0, 1, \ldots$ there exist trigonometric polynomials $q_0, \ldots, q_n$ with mutually disjoint spectra, a measure preserving bijection τ of $[0,1]$, and a set $B \subset [0,1]$ with measure $1-\delta$ such that $|v_k(\tau(t)) - q_k(t)| < \delta$ for $t \in B,\ k = 0, 1, \ldots, n$. Moreover, we can choose these polynomials with $\|q_k\|_\infty \leq 1$.*

Proof. Recall that in Lemma 5.2 we have already constructed trigonometric polynomials $p_0, \ldots, p_{m-1}$ with mutually disjoint spectra and a set $A \subset [0,1]$ with measure $1 - \varepsilon$ such that $|\chi_h(t) - p_h(t)| < \varepsilon$ for $t \in A$ and $h = 0, 1, \ldots, m-1$. In addition, we have that $\operatorname{spec}(p_h) \subset m\mathbb{Z} + h$. Denote by N_h the degree of the polynomial p_h and put $N = \max N_h$. Let K be an integer with $m^K > 2N$. For $k = \sum_{j=0}^{n-1} b_j m^j$ with $b_j \in \{0, 1, \dots, m-1\}$ we define

$$q_k(t) = \prod_{j=0}^{n-1} p_{b_j}(m^{Kj}t) \ \text{ and } \ \widetilde{v}_k(t) = \prod_{j=0}^{n-1} s_{Kj}(t)^{b_j} = \prod_{j=0}^{n-1} s_0(m^{Kj}t)^{b_j} = \prod_{j=0}^{n-1} \chi_{b_j}(m^{Kj}t).$$

Since $\|p_h\|_\infty \leq 1$ for every $h = 0, \dots, m-1$, we obtain $\|q_k\|_\infty \leq 1$. It is easy to see that there exists a measure preserving bijection τ of the interval $[0,1]$ such that $s_j(\tau(t)) = s_{Kj}(t)$ for $j = 0, 1, \dots, n-1$ and $t \in [0,1]$. Now we have $v_k(\tau(t)) = \widetilde{v}_k(t)$ for $k = 0, \dots, m^n - 1$. Moreover there is a set $B \subset [0,1)$ with measure $1 - \delta$ such that for $t \in B$

$$\begin{aligned} |\widetilde{v}_k(t) - q_k(t)| &= \left| \prod_{j=0}^{n-1} \chi_{b_j}(m^{Kj}t) - \prod_{j=0}^{n-1} p_{b_j}(m^{Kj}t) \right| \\ &\leq \sum_{h=0}^{n-1} \left| \prod_{j=0}^{h-1} \chi_{b_j}(m^{Kj}t) \prod_{j=h}^{n-1} p_{b_j}(m^{Kj}t) - \prod_{j=0}^{h} \chi_{b_j}(m^{Kj}t) \prod_{j=h+1}^{n-1} p_{b_j}(m^{Kj}t) \right| \\ &\leq \sum_{h=0}^{n-1} |\chi_{b_h}(m^{Kh}t) - p_{b_h}(m^{Kh}t)| < n\varepsilon. \end{aligned}$$

Taking $\delta = n\varepsilon$ gives the trigonometric polynomials $q_0, \dots, q_{m^n-1}$ such that $|v_k(\tau(t)) - q_k(t)| < \delta$ for $t \in B$. The task is now to show that the constructed polynomials have mutually disjoint spectra. From the definition of q_k we obtain

$$\operatorname{spec}(q_k) \subset \left\{ \sum_{j=0}^{n-1} \beta_j m^{Kj} : \beta_j \in \operatorname{spec}(p_{b_j}) \right\}.$$

Let us assume that there are $k = \sum_{j=0}^{n-1} b_j m^j$ and $k' = \sum_{j=0}^{n-1} b'_j m^j$ with $b_j, b'_j \in \{0, 1, \dots, m-1\}$ such that

$$\operatorname{spec}(q_k) \cap \operatorname{spec}(q_{k'}) \neq \emptyset$$

Then we have

$$\sum_{j=0}^{n-1} \beta_j m^{Kj} = \sum_{j=0}^{n-1} \beta'_j m^{Kj}$$

for some $\beta_j, \beta_j^{'} \in \{k \in \mathbb{Z} : |k| < N\}$. First observe that it follows from $\sum_{j=0}^{n-1} a_j m^{Kj} = 0$ with integers a_j satisfying $|a_j| < m^K$ that $a_j = 0$ for $j = 0, 1, \ldots, n-1$. Consequently, we obtain from $|\beta_j - \beta_j'| \leq 2N < m^K$ that

$$\beta_j = \beta_j' \ \text{ for } \ j = 0, 1, \ldots, n-1.$$

If $k \neq k'$ this contradicts the fact that $\text{spec}(p_k) \cap \text{spec}(p_{k'}) = \emptyset$. This shows that $k = k^{'}$ which finishes the proof. $\square$

5.3 The general case

Let $C_c^X(H)$ be the space of all compactly supported X-valued functions on the lca group H. In order to prove Theorem 5.1 in the case that G is compact or discrete the following two results are needed. The first one is due to M.E. Andersson (see [And98], Lemma 3.2 and Proposition 3.3).

PROPOSITION 5.4. *Let H be an open subgroup of an lca group G and let $g \in C_c^X(H)$. Let f be the extension of g to all of G by zero on $G \setminus H$. Then*

$$\frac{\|(\mathcal{F}_G \otimes T)g\|_{L_{p'}^Y(H')}}{\|g\|_{L_p^X(H)}} = \frac{\|(\mathcal{F}_H \otimes T)f\|_{L_{p'}^Y(G')}}{\|f\|_{L_p^X(G)}}.$$

In particular, $\|T|\mathcal{FT}_p^H\| \leq \|T|\mathcal{FT}_p^G\|$.

The second one is taken from [HL02].

PROPOSITION 5.5. *Let G_1, G_2, H be lca groups and let $1 < p \leq 2$. If all $T \in \mathcal{FT}_p^{G_2}$ satisfy $\|T|\mathcal{FT}_p^{G_1}\| \leq \|T|\mathcal{FT}_p^{G_2}\|$, then also $\|T|\mathcal{FT}_p^{G_1 \times H}\| \leq \|T|\mathcal{FT}_p^{G_2 \times H}\|$ for all $T \in \mathcal{FT}_p^{G_2 \times H}$.*

The proof of this interesting transference principle is based on the following crucial observation from [And98] which connects the Fourier type of an operator T with respect to the product group $G \times H$ with the Fourier type of the tensor product $\mathcal{F}_H \otimes T$, see also [Hin03a].

PROPOSITION 5.6. *Let G and H be lca groups and $1 < p \leq 2$. Then $T \in \mathcal{FT}_p^{G \times H}$ if and only if $T \in \mathcal{FT}_p^H$ and $\mathcal{F}_H \otimes T : L_p^X(H) \to L_{p'}^Y(H') \in \mathcal{FT}_p^G$. Moreover in this case*

$$\|\mathcal{F}_H \otimes T|\mathcal{FT}_p^G\| = \|T|\mathcal{FT}_p^{G\times H}\|.$$

Now we are in a position to prove our main result in the case that G is compact or discrete.

PROPOSITION 5.7. *Let G be a compact or discrete abelian group. If an operator T has Fourier type 2 with respect to the classical groups, then it has also Fourier type 2 with respect to G. Moreover,*

$$\|T|\mathcal{FT}_2^G\| \leq \|T|\mathcal{FT}_2^{\mathbb{T}}\|.$$

Proof. A basic duality result shows that $\|T|\mathcal{FT}_2^G\| = \|T'|\mathcal{FT}_2^{G'}\|$ for all operators T and all lca groups G, see Section 2.3 and [HL02] for a better duality result. Since dual groups of compact groups are discrete, equation (5.3) implies that we may assume that G is discrete. By density, it is enough to prove that for any function g with finite support on G the following inequality holds

$$\|(\mathcal{F}_G \otimes T)g\|_{L_2^Y(G')} \leq \|T|\mathcal{FT}_2^{\mathbb{T}}\| \, \|g\|_{L_2^X(G)}. \tag{5.8}$$

Any such function g with finite support $\operatorname{supp}(g) \subset G$ has the form

$$g(s) = x_s \in X, \ \ s \in \ \operatorname{supp}(g) \quad \text{and} \quad g|_{G\setminus \operatorname{supp}(g)} \equiv 0.$$

The finitely generated group $H = \langle \operatorname{supp}(g)\rangle$ is topologically isomorphic to the product $\mathbb{Z}^n \times F$, where $n \in \mathbb{N}_0$ and F is a suitable finite abelian group. Letting $f := g|_H$, it follows by Proposition 5.4 that (5.8) is equivalent to

$$\|(\mathcal{F}_H \otimes T)f\|_{L_2^Y(H')} \leq \ \|T|\mathcal{FT}_2^{\mathbb{T}}\| \, \|f\|_{L_2^X(H)}. \tag{5.9}$$

The definition of Fourier type yields

$$\|(\mathcal{F}_H \otimes T)f\|_{L_2^Y(H')} \leq \ \|T|\mathcal{FT}_2^{H}\| \, \|f\|_{L_2^X(H)}. \tag{5.10}$$

We obtain from (5.3) and Proposition 5.5 that

$$\|T|\mathcal{FT}_2^{\mathbb{Z}^n\times F}\| = \|T|\mathcal{FT}_2^{\mathbb{Z}\times F}\|.$$

Since F is finite, the result from the previous section and (5.3) again yield

$$\|T|\mathcal{FT}_2^F\| \leq \|T|\mathcal{FT}_2^{\mathbb{T}}\| = \|T|\mathcal{FT}_2^{\mathbb{Z}}\|,$$

from which we now conclude that

$$\|T|\mathcal{FT}_2^{\mathbb{Z}\times F}\| \leq \|T|\mathcal{FT}_2^{\mathbb{Z}\times\mathbb{Z}}\| = \|T|\mathcal{FT}_2^{\mathbb{Z}}\| = \|T|\mathcal{FT}_2^{\mathbb{T}}\|.$$

Altogether, we find that

$$\|T|\mathcal{FT}_2^H\| = \|T|\mathcal{FT}_2^{\mathbb{Z}^n\times F}\| \leq \|T|\mathcal{FT}_2^{\mathbb{T}}\|$$

which shows that (5.10) indeed implies (5.9) completing the proof. □

We are now in a position to establish the proof of our main result.

Proof of Theorem 5.1. We divide the proof into three steps.
First step. Here we consider the case that G is isomorphic to $\mathbb{Z}^n \times K$ for some $n \in \mathbb{N}_0$ and some compact abelian group K. By Propositions 5.5 and 5.7 and (5.3) we obtain

$$\|T|\mathcal{FT}_2^G\| = \|T|\mathcal{FT}_2^{\mathbb{Z}^n\times K}\| = \|T|\mathcal{FT}_2^{\mathbb{Z}\times K}\| \leq \|T|\mathcal{FT}_2^{\mathbb{Z}\times\mathbb{T}}\| = \|T|\mathcal{FT}_2^{\mathbb{T}}\|.$$

Second step. Let now G be an lca group with an open and compact subgroup H. Consider the canonical quotient map $q : G \to G/H$. The quotient G/H is a discrete abelian group. Again a density argument shows that it is sufficient to prove that

$$\|(\mathcal{F}_G \otimes T)f\|_{L_2^Y(G')} \leq \ \|T|\mathcal{FT}_2^{\mathbb{T}}\| \, \|f\|_{L_2^X(G)}, \tag{5.11}$$

for functions f vanishing on $G \setminus q^{-1}(S)$ for some finite subset $S \subset G/H$. Let $M = q^{-1}(\langle S\rangle)$ be the preimage of the finitely generated group $\langle S\rangle$. Since $\langle S\rangle$ is topologically isomorphic with $\mathbb{Z}^n \times F$ for some $n \in \mathbb{N}_0$ and some finite abelian group F, the group M is topologically isomorphic to $\mathbb{Z}^n \times K$ for some suitable compact group K. Let $g := f|_M$. The first step yields

$$\|(\mathcal{F}_M \otimes T)g\|_{L_2^Y(M')} \leq \ \|T|\mathcal{FT}_2^{\mathbb{T}}\| \, \|g\|_{L_2^X(M)}.$$

Now an appeal to Proposition 5.4 proves (5.11).

Third step. Finally, we consider the case of a general lca group G. The structure theorem for lca groups tells us that G is topologically isomorphic with $\mathbb{R}^n \times H$ for some $n \in \mathbb{N}_0$ and some lca group H having a compact and open subgroup. Proposition 5.5 and (5.3) together with the second step yield

$$\|T|\mathcal{FT}_2^G\| = \|T|\mathcal{FT}_2^{\mathbb{R}^n\times H}\| \leq \|T|\mathcal{FT}_2^{\mathbb{T}^n\times H}\| = \|T|\mathcal{FT}_2^{\mathbb{T}\times H}\| \leq \|T|\mathcal{FT}_2^{\mathbb{T}}\|.$$

This gives the claim also in this case. $\square$

Chapter 6

Absolutely continuous operators

6.1 Introduction

The purpose of this introductory section is to motivate for studying the notion of absolute continuity in the setting of operators between Banach spaces. Let Ω be a compact topological space. Consider a weakly compact operator T defined on the space $C(\Omega)$ consisting of all real valued continuous functions on Ω with values in some Banach space X. A result of R.G. Bartle, N. Dunford and J. Schwarz from [BDS55] asserts that there exists a certain control measure μ for the operator T. More precisely, it can be shown that for every $\varepsilon > 0$ there exists a positive constant $N(\varepsilon)$ such that

$$\|Tf\| \leq N(\varepsilon) \int |f| d\mu + \varepsilon \|f\| \quad \text{for every} \quad f \in C(\Omega).$$

Motivated by the above property C. P. Niculescu introduced in his pioneering work [Nic75] and [Nic79] the class of absolutely continuous operators with respect to a certain seminorm. A systematic study of this notion was initiated in the work of H. Jarchow, U. Matter [Mat87, Mat89, JM88] and F. Räbiger [Räb91].

In order to generalize the ideas of Niculescu the following definition was considered in [Mat87]. Let $T \in \mathcal{L}(X, Y)$, $S \in \mathcal{L}(X, Z)$ and $R \in \mathcal{L}(X, W)$ be operators between Banach spaces X, Y, Z and W. An operator T is said to be *absolutely continuous with respect to* (S, R), denoted by $T \ll (S, R)$, if for arbitrary $\varepsilon > 0$ there is a constant $N(\varepsilon) \geq 0$ such that

$$\|Tx\| \leq N(\varepsilon)\|Sx\| + \varepsilon\|Rx\| \quad \text{for all } x \in X. \tag{6.1}$$

In the case when $X = W$ and $R = I_X$, we will call such an operator T absolutely continuous with respect to S and denote this by $T \ll S$. It was shown by C. P. Niculescu that $T \ll S$ if and only if $T'' \ll S''$. Let us now consider the notion of absolute continuity of operators from the operator ideal point of view. For definitions and facts from operator ideal theory we refer the reader to Section 2.2. The following result of H. Jarchow and A. Pełczyński characterizes the closed injective hull $\overline{\mathcal{A}}^{\text{inj}}$ of $\mathcal{A}$, see [Jar81].

THEOREM 6.1. *Let $\mathcal{A}$ be a quasinormed operator ideal. An operator $T \in \mathcal{L}(X,Y)$ belongs to the closed injective hull $\overline{\mathcal{A}}^{\text{inj}}$ of $\mathcal{A}$ if and only if there exist a Banach space Z and an operator $S \in \mathcal{A}(X,Z)$ such that $T \ll S$.*

We conclude this section by stressing an important connection of this notion with interpolation theory. Let us consider for a fixed $r > 0$ the function $N(\varepsilon) = \varepsilon^{-r}$ as a function appearing in (6.1). More precisely, for a fixed $\theta \in (0,1)$ and $x \in X$ computing the minimum value of the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ given by $f(t) = t^{\theta/(\theta-1)}\|Sx\| + t\|Rx\|$, which controls the right hand side of inequality (6.1), shows that the definition of absolute continuity of operators is equivalent to the following statement:

$$\|Tx\| \leq \|\widetilde{S}x\|^{1-\theta}\|Rx\|^{\theta} \quad \text{for all } x \in X. \tag{6.2}$$

Here $\widetilde{S}$ denotes a constant multiple of the operator S occurring in (6.1). The value of the underlying constant is $((1-\theta)/\theta)^{\theta} + (\theta/(1-\theta))^{\theta}$. The inequalities of type (6.2) play an important rôle in interpolation theory, see [BK91]. This problem will be discussed in detail in Section 6.3.

Let us now consider the function $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$ given by $\varphi(s,t) = s^{1-\theta}t^{\theta}$. Then the condition (6.2) reads as follows

$$\|Tx\| \leq \varphi(\|Sx\|, \|Rx\|) \quad \text{for all } x \in X. \tag{6.3}$$

For simplicity of notation, we write here S instead of $\widetilde{S}$. Interesting questions that arise from the above considerations are the following: For which functions

$\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$ does (6.3) already imply $T \ll (S,R)$, and conversely, does the existence of a function φ such that (6.3) holds already follows from $T \ll (S,R)$?
To answer this questions let us define the class **AC** consisting of all continuous, positively homogeneous, concave functions $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$ with $\varphi(0,0) = 0, \varphi(s,0) = \varphi(0,t) = 0$ and $\varphi(1,1) = 1$. Taking into account the homogeneity of φ we put often $\varphi(s,t) = s\rho(t/s)$ with $\rho : [0,\infty) \to \mathbb{R}_+$. The function ρ is also non-decreasing, concave and continuous with $\rho(0) = 0$ and $\rho(1) = 1$. It was shown by F. Räbiger in [Räb91] that the following statements are equivalent

- $T \ll (S,R)$.
- There exists a function $\varphi \in$ **AC** such that (6.3) holds.

6.2 An interpolative ideal procedure

This section presents a procedure by which, from given operator ideals $\mathcal{A}$ and $\mathcal{B}$, a scale of new ideals $(\mathcal{A},\mathcal{B})_\varphi$ is generated. For definitions and basic facts on operator ideals we refer the reader to Section 2.2. Recall that $\mathcal{L}$ denotes the ideal of all operators between arbitrary Banach spaces. In what follows, let $\varphi \in$ **AC**. We start our considerations by showing the superadditivity of φ. This property will be frequently used in the sequel.

LEMMA 6.2. *For any $a_i, b_i \geq 0$ the following inequality holds*

$$\sum_{i=1}^n \varphi(a_i,b_i) \leq \varphi\left(\sum_{i=1}^n a_i, \sum_{i=1}^n b_i\right). \tag{6.4}$$

Proof. By the definition of concavity the inequality

$$\sum_{i=1}^n \lambda_i\varphi(a_i,b_i) \leq \varphi\left(\sum_{i=1}^n \lambda_i a_i, \sum_{i=1}^n \lambda_i b_i\right)$$

holds for every $\lambda_i \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$. The homogeneity of φ gives

$$\sum_{i=1}^n c\lambda_i\varphi(a_i,b_i) \leq \varphi\left(\sum_{i=1}^n c\lambda_i a_i, \sum_{i=1}^n c\lambda_i b_i\right) \quad \text{for every } c > 0.$$

Taking $c\lambda_i = 1$ for every $i = 1,\dots,n$ gives the claim.

$\square$

The following notion gives a generalization of the procedure introduced by U. Matter in [Mat87]. An operator $T \in \mathcal{L}(X,Y)$ belongs to $(\mathcal{A},\mathcal{B})_\varphi$ if there exist Banach spaces Z, W and operators $S \in \mathcal{A}(X,Z)$, $R \in \mathcal{B}(X,W)$ such that

$$\|Tx\| \leq \varphi(\|Sx\|, \|Rx\|) \quad \text{for all } \ x \in X. \tag{6.5}$$

It is easy to see that the above defined class possesses the ideal property. To show that $TV \in (\mathcal{A},\mathcal{B})_\varphi(X_0,Y)$ for $V \in \mathcal{L}(X_0,X)$ and $T \in (\mathcal{A},\mathcal{B})_\varphi(X,Y)$ we choose operators $S \in \mathcal{A}(X,Z)$, $R \in \mathcal{B}(X,W)$ according to (6.5). Then $SV \in \mathcal{A}(X_0,Z)$ and $RV \in \mathcal{B}(X_0,W)$. We check at once that

$$\|TVx\| \leq \varphi(\|SVx\|, \|RVx\|) \quad \text{for all } \ x \in X_0.$$

To deduce that $UT \in (\mathcal{A},\mathcal{B})_\varphi(X,Y_0)$ for $U \in \mathcal{L}(Y,Y_0)$ and $T \in (\mathcal{A},\mathcal{B})_\varphi(X,Y)$ let us take the $\|U\|$-multiple of operators appearing in (6.5). By homogeneity of φ we obtain

$$\|UTx\| \leq \|U\|\|Tx\| \leq \|U\|\varphi\left(\|Sx\|, \|Rx\|\right) = \varphi\left(\|\|U\|Sx\|, \|\|U\|Rx\|\right) \quad \text{for all } x \in X.$$

In order to see that $(\mathcal{A},\mathcal{B})_\varphi$ is a linear space, we provide the following alternative characterization of operators for this class.

PROPOSITION 6.3. *An operator T belongs to $(\mathcal{A},\mathcal{B})_\varphi$ if and only if there exist some $n \in \mathbb{N}$, Banach spaces Z_i, W_i and operators $S_i \in \mathcal{A}(X,Z_i), R_i \in \mathcal{B}(X,W_i), i = 1,2,\ldots,n$, such that*

$$\|Tx\| \leq \sum_{i=1}^{n} \varphi(\|S_i x\|, \|R_i x\|) \quad \textit{for all } \ x \in X. \tag{6.6}$$

Proof. The only if part is obvious. In order to prove the converse implication let

$$Z := \Big(\bigoplus_{i=1}^{n} Z_i\Big)_1 \text{ and } \ W := \Big(\bigoplus_{i=1}^{n} W_i\Big)_1.$$

We define operators $S : X \to Z$ and $R : X \to W$ by

$$S := \sum_{i=1}^{n} J_{Z_i}^{Z} S_i \quad \text{ and } \quad R := \sum_{i=1}^{n} J_{W_i}^{W} R_i,$$

where $J^Z_{Z_i} : Z_i \to Z$ denotes the canonical injection. By lemma 6.2 we obtain

$$\|Tx\| \leq \sum_{i=1}^n \varphi(\|S_ix\|, \|R_ix\|) \leq \varphi\left(\sum_{i=1}^n \|S_ix\|, \sum_{i=1}^n \|R_ix\|\right) = \varphi(\|Sx\|, \|Rx\|),$$

which finishes our proof. $\square$

Now it follows from Proposition 6.3 that with operators $T_1, T_2 \in (\mathcal{A}, \mathcal{B})_\varphi$ and numbers λ_1, λ_2 also $\lambda_1 T_1 + \lambda_2 T_2 \in (\mathcal{A}, \mathcal{B})_\varphi$. Hence $(\mathcal{A}, \mathcal{B})_\varphi$ is an operator ideal.
From now on, we assume that α, β are quasinorms on $\mathcal{A}, \mathcal{B}$, such that $\mathcal{A}$ and $\mathcal{B}$, respectively, become quasinormed Banach ideals. We now consider the following maps, which are connected to part (6.5) and (6.6), respectively: For each operator $T \in (\mathcal{A}, \mathcal{B})_\varphi$ we put

(i)
$$\gamma(T) := \inf \varphi(\alpha(S), \beta(R)),$$
where the infimum ranges over all operators S, R such that inequality (6.5) holds.

(ii)
$$\overline{\gamma}(T) := \inf \sum_{i=1}^n \varphi(\alpha(S_i), \beta(R_i)),$$
where the infimum ranges over all $n \in \mathbb{N}$ and all operators S_i, R_i such that inequality (6.6) holds.

We recall that a mapping $\alpha : \mathcal{A} \to \mathbb{R}_+$ (in particular we can consider a quasinorm) is said to have the *ideal property* if for $V \in \mathcal{L}(X_0, X), T \in \mathcal{A}(X, Y)$ and $U \in \mathcal{L}(Y, Y_0)$ the following inequality holds

$$\alpha(UTV) \leq \|U\|\alpha(T)\|V\|.$$

PROPOSITION 6.4. *Both maps γ and $\overline{\gamma}$ possess the ideal property. Moreover, the map $\overline{\gamma}$ is a norm on $(\mathcal{A}, \mathcal{B})_\varphi$.*

Proof. We prove the first statement only for γ. The proof for $\overline{\gamma}$ is similar. Let $V \in \mathcal{L}(X_0, X)$ and $U \in \mathcal{L}(Y, Y_0)$. Let S, R be operators such that

$$\|Tx\| \leq \varphi(\|Sx\|, \|Rx\|)$$

holds. Observe that

$$\|(UTV)x\| \leq \|U\|\|T(Vx)\| \leq \|U\|\varphi(\|S(Vx)\|, \|R(Vx)\|)$$
$$= \varphi(\|\|U\|SVx\|, \|\|U\|RVx\|).$$

So $\widetilde{S} := \|U\|SV$ and $\widetilde{R} := \|U\|RV$ are admissible operators in the definition of $\gamma(UTV)$. We obtain

$$\gamma(UTV) \leq \varphi(\alpha(\|U\|SV), \beta(\|U\|RV)) \leq \|U\|\varphi(\alpha(S), \beta(R))\|V\|.$$

Taking the infimum over all operators S and R gives the claim.
We only have to show the triangle inequality in the second statement. For that reason, let $T_1, T_2 \in (\mathcal{A}, \mathcal{B})_\varphi$ and assume that $S_1, \dots, S_m; R_1, \dots, R_m$ are such that

$$\|T_1 x\| \leq \sum_{i=1}^{n} \varphi(\|S_i x\|, \|R_i x\|) \text{ and } \|T_2 x\| \leq \sum_{i=n+1}^{m} \varphi(\|S_i x\|, \|R_i x\|) \text{ for some } n < m.$$

Obviously

$$\|T_1 x + T_2 x\| \leq \sum_{i=1}^{m} \varphi(\|S_i x\|, \|R_i x\|).$$

We then have

$$\overline{\gamma}(T_1 + T_2) \leq \sum_{i=1}^{n} \varphi(\alpha(S_i), \beta(R_i)) + \sum_{i=n+1}^{m} \varphi(\alpha(S_i), \beta(R_i))$$

Turning to the infimum on the right hand side gives the claim. □

THEOREM 6.5. *Let ρ be a submultiplicative function, i.e. there exists constant $c > 0$ such that*

$$\rho(st) \leq c\rho(s)\rho(t) \text{ for every } s, t \in \mathbb{R}_+.$$

Assume also that α, β are ideal norms. Then

$$\overline{\gamma}(T) \leq \gamma(T) \leq c\,\overline{\gamma}(T). \tag{6.7}$$

In other words, both maps are equivalent provided that the function ρ is submultiplicative.

Proof. The left hand inequality in (6.7) is obvious. First we prove that the following inequality holds for any $\xi, \eta, \tau \in \mathbb{R}_+$:

$$\varphi(\xi, \eta) \leq c\, \varphi\Big(\varphi(1, \tau)\xi, \varphi(1/\tau, 1)\eta\Big). \tag{6.8}$$

This inequality is equivalent to

$$\xi\rho\Big(\frac{\eta}{\xi}\Big) \leq c\, \varphi(1, \tau)\xi\, \rho\left(\frac{\varphi(1/\tau, 1)\eta}{\varphi(1, \tau)\xi}\right)$$

which follows from the submultiplicativity of ρ by

$$\rho\left(\frac{\eta}{\xi}\right) \leq c\, \rho(\tau)\rho\left(\frac{1}{\tau}\frac{\eta}{\xi}\right).$$

Now let operators $S_i \in \mathcal{A}(X, Z_i), R_i \in \mathcal{B}(X, W_i), i = 1, 2, \dots, n$ be such that

$$\|Tx\| \leq \sum_{i=1}^{n} \varphi(\|S_i x\|, \|R_i x\|).$$

For the proof of the second inequality in (6.7) let us define

$$\xi_i := \|S_i x\|, \;\; \eta_i := \|R_i x\|, \;\; \tau_i := \frac{\beta(R_i)}{\alpha(S_i)}.$$

Furthermore, let

$$Z = \Big(\bigoplus_{i=1}^{n} Z_i\Big)_1 \quad \text{and} \quad W = \Big(\bigoplus_{i=1}^{n} W_i\Big)_1.$$

and define $S \in \mathcal{A}(X, Z), R \in \mathcal{B}(X, W)$ by

$$S := \sum_{i=1}^{n} \rho(\tau_i) J_{Z_i}^{Z} S_i \quad \text{and} \quad R := \sum_{i=1}^{n} \frac{1}{\tau_i}\rho(\tau_i) J_{W_i}^{W} R_i.$$

Then we have

$$\|Sx\| = \sum_{i=1}^{n} \rho(\tau_i)\|S_i x\| \quad \text{and} \quad \|Rx\| = \sum_{i=1}^{n} \frac{1}{\tau_i}\rho(\tau_i)\|R_i x\|.$$

Using Lemma 6.2 and the inequality (6.8) yields

$$
\begin{aligned}
\|Tx\| &\leq \sum_{i=1}^{n} \varphi(\|S_i x\|, \|R_i x\|) = \sum_{i=1}^{n} \varphi(\xi_i, \eta_i) \leq c \sum_{i=1}^{n} \varphi\Big(\rho(\tau_i)\xi_i, \frac{\rho(\tau_i)}{\tau_i}\eta_i\Big) \\
&\leq c\varphi\Big(\sum_{i=1}^{n} \rho(\tau_i)\xi_i, \sum_{i=1}^{n} \frac{\rho(\tau_i)}{\tau_i}\eta_i\Big) = c\varphi(\|Sx\|, \|Rx\|).
\end{aligned}
$$

Furthermore, we obtain

$$
\begin{aligned}
\varphi(\alpha(S), \beta(R)) &\leq \varphi\Big(\sum_{i=1}^{n} \rho(\tau_i)\alpha(S_i), \sum_{i=1}^{n} \frac{\rho(\tau_i)}{\tau_i}\beta(R_i)\Big) \\
&= \sum_{i=1}^{n} \alpha(S_i)\rho(\tau_i)\varphi(1,1) = \sum_{i=1}^{n} \varphi(\alpha(S_i), \beta(R_i)),
\end{aligned}
$$

which finishes the proof. □

To show the completeness of $(\mathcal{A}, \mathcal{B})_\varphi$ we consider $(T_n) \in (\mathcal{A}, \mathcal{B})_\varphi(X, Y)$ such that $\sum_{n=1}^{\infty} \gamma(T_n) < \infty$. Our aim is to find an operator $T \in (\mathcal{A}, \mathcal{B})_\varphi(X, Y)$ such that $\gamma\left(T - \sum_{i=1}^{n} T_i\right)$ tends to zero as $n \to \infty$. By assumption there are Banach spaces Z_i, W_i and operators S_i, R_i such that $\|T_i x\| \leq \varphi(\|S_i x\|, \|R_i x\|)$ for all $x \in X$. Moreover, we obtain that $\sum_{i=1}^{\infty} \alpha(S_i)$ and $\sum_{i=1}^{\infty} \beta(R_i)$ are finite. Put $Z = \Big(\bigoplus_{i=1}^{\infty} Z_i\Big)_1$ and $W = \Big(\bigoplus_{i=1}^{\infty} W_i\Big)_1$. Now the completeness of $\mathcal{A}$ and $\mathcal{B}$ yields

$$
\alpha\Big(S - \sum_{i=1}^{n} S_i\Big) \overset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \beta\Big(R - \sum_{i=1}^{n} R_i\Big) \overset{n\to\infty}{\longrightarrow} 0
$$

with $S = \Big(\bigoplus_{i=1}^{\infty} S_i\Big)_1$ and $R = \Big(\bigoplus_{i=1}^{\infty} S_i\Big)_1$. Consequently, we obtain for $T = \Big(\bigoplus_{i=1}^{\infty} T_i\Big)_1$ that

$$
\gamma\Big(T - \sum_{i=1}^{n} T_i\Big) \leq \varphi\Big(\alpha\Big(S - \sum_{i=1}^{n} S_i\Big), \beta\Big(R - \sum_{i=1}^{n} R_i\Big)\Big).
$$

Moreover we have

$$
\begin{aligned}
\Big\|(T - \sum_{i=1}^{n} T_i)x\Big\| &\leq \sum_{i=n+1}^{\infty} \varphi\left(\|S_i x\|, \|R_i x\|\right) \leq \varphi\Big(\sum_{i=n+1}^{\infty} \|S_i x\|, \sum_{i=n+1}^{\infty} \|R_i x\|\Big) \\
&= \varphi\Big(\Big\|(S - \sum_{i=1}^{n} S_i)x\Big\|, \Big\|(R - \sum_{i=1}^{n} R_i)x\Big\|\Big).
\end{aligned}
$$

This shows our assertion.
We have thus proved that if ρ is submultiplicative then $[(\mathcal{A},\mathcal{B})_\varphi,\overline{\gamma}]$ is a Banach ideal and γ is equivalent to the norm $\overline{\gamma}$. We collect the results obtained so far in the following theorem.

THEOREM 6.6. *Let $\varphi \in \boldsymbol{AC}$ and $\rho : [0,\infty) \to \mathbb{R}_+$ be given by $\varphi(s,t) = s\rho(t/s)$. Let $\mathcal{A}$ and $\mathcal{B}$ be operators ideals. Then $(\mathcal{A},\mathcal{B})_\varphi$ is an operator ideal. If, moreover, $(\mathcal{A},\alpha)$, $(\mathcal{B},\beta)$ are quasinormed Banach ideals and ρ is submultiplicative, then $[(\mathcal{A},\mathcal{B})_\varphi,\overline{\gamma}]$ is a Banach ideal where $\overline{\gamma}$ is given by (ii)*

Straightforward computation yields the following reiteration property.

PROPOSITION 6.7. *Let $\varphi,\varphi_0,\varphi_1 \in$ **AC**. Then*

$$\big((\mathcal{A},\mathcal{B})_{\varphi_0},(\mathcal{A},\mathcal{B})_{\varphi_1}\big)_\varphi \subset (\mathcal{A},\mathcal{B})_{\varphi(\varphi_0,\varphi_1)}.$$

Proof. Take $T \in \big((\mathcal{A},\mathcal{B})_{\varphi_0},(\mathcal{A},\mathcal{B})_{\varphi_1}\big)_\varphi(X,Y)$. Then we find Banach spaces Z,W and operators $T_0 \in (\mathcal{A},\mathcal{B})_{\varphi_0}(X,Z)$ and $T_1 \in (\mathcal{A},\mathcal{B})_{\varphi_1}(X,W)$ such that

$$\|Tx\| \le \varphi(\|T_0x\|,\|T_1x\|) \quad \text{for all} \;\; x \in X.$$

By definition, for $T_i \in (\mathcal{A},\mathcal{B})_{\varphi_i}(X,Y),\ i=0,1$, we find Banach spaces Z_i,W_i and operators $S_i \in \mathcal{A}(X,Z_i)$ and $R_i \in \mathcal{B}(X,W_i)$ such that

$$\|T_ix\| \le \varphi_i(\|S_ix\|,\|R_ix\|) \quad \text{for all} \;\; x \in X, \;\; i=0,1.$$

Define $S = S_0 \oplus_1 S_1$ and $S = R_0 \oplus_1 R_1$. By the monotonicity of φ,φ_0 and φ_1 we obtain

$$\begin{aligned}\|Tx\| \le \varphi(\|T_0x\|,\|T_1x\|) &\le \varphi\left(\varphi_0(\|S_0x\|,\|R_0x\|),\varphi_1(\|S_1x\|,\|R_1x\|)\right)\\ &\le \varphi\left(\varphi_0(\|Sx\|,\|Rx\|),\varphi_1(\|Sx\|,\|Rx\|)\right).\end{aligned}$$

This shows that $T \in (\mathcal{A},\mathcal{B})_{\varphi(\varphi_0,\varphi_1)}(X,Y)$, which completes the proof. □

6.3 Factoring through interpolation spaces

Let us start this section by recalling some basic notation and results from interpolation theory. A pair of Banach spaces (X_0,X_1) is said to be an *interpolation* *couple* (or compatible couple) if both spaces are continuously embedded into a certain Hausdorff topological vector space V. For an interpolation couple (X_0, X_1) we put

$$\Delta(X_0, X_1) = \{x \in X_0 \cap X_1 : \ \|x\|_\Delta = \max(\|x_0\|_{X_0}, \|x_1\|_{X_1}) < \infty\},$$
$$\Sigma(X_0, X_1) = \{x = x_0 + x_1 \in X_0 + X_1 : \ \|x\|_\Sigma = \inf_{x=x_0+x_1} \{\|x_0\|_{X_0} + \|x_1\|_{X_1}\} < \infty\}.$$

The linear spaces $\Delta(X_0, X_1)$ and $\Sigma(X_0, X_1)$ equipped with the norms $\|\cdot\|_\Delta$ and $\|\cdot\|_\Sigma$, respectively are Banach spaces. A Banach space X is said to be an *intermediate space* with respect to (X_0, X_1) if $\Delta(X_0, X_1) \hookrightarrow X \hookrightarrow \Sigma(X_0, X_1)$ (continuous inclusion). If additionally for every linear operator $T : X_0 + X_1 \to X_0 + X_1$ such that the restrictions $T_0 : X_0 \to X_0$ and $T_1 : X_1 \to X_1$ are bounded we have that $T : X \to X$ is bounded, then we refer X to as *interpolation space* with respect to (X_0, X_1). Let (X_0, X_1) and (Y_0, Y_1) be interpolation couples. From now on, the notation $T : (X_0, X_1) \to (Y_0, Y_1)$ means that $T : \Sigma(X_0, X_1) \to \Sigma(Y_0, Y_1)$ is a linear operator such that the restrictions of T given by $T_0 : X_0 \to X_1$ and $T_1 : Y_0 \to Y_1$ are bounded. A functor $\mathcal{F}$ from the category of all compatible couples into the category of all Banach spaces is called *interpolation functor* (or *interpolation method*) if for any couple (X_0, X_1) the Banach space $\mathcal{F}(X_0, X_1)$ is an intermediate space and $T : \mathcal{F}(X_0, X_1) \to \mathcal{F}(Y_0, Y_1)$ is bounded for all couples (X_0, X_1), (Y_0, Y_1) and any $T : (X_0, X_1) \to (Y_0, Y_1)$. The closed graph theorem implies that for any interpolation functor $\mathcal{F}$ there exists a constant $C > 0$ such that we have

$$\|T : \mathcal{F}(X_0, X_1) \to \mathcal{F}(Y_0, Y_1)\| \le C \max\Big(\|T : X_0 \to Y_0\|, \|T : X_1 \to Y_1\|\Big).$$

If C can be chosen equal to one then we say that $\mathcal{F}$ is an *exact* interpolation functor. Fundamental examples of the exact interpolation methods are the *real method of interpolation* $(\cdot, \cdot)_{\theta,p}$ with $0 < \theta < 1$ and $1 \le p \le \infty$ which takes its origins from the classical Marcinkiewicz theorem and the *complex method of interpolation* $[\cdot, \cdot]_\theta$. The idea of this method goes back to the Riesz-Thorin theorem. The most important notion of real interpolation theory is the K-functional. Let (X_0, X_1) be a Banach couple. The K-functional is given by

$$K(x, t) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : \ \ x = x_0 + x_1\} \text{ for } \ x \in \Sigma(X_0, X_1) \text{ and } t > 0.$$

Thus $K(x,1)$ is simply the usual norm of the sum space. Moreover the mapping $x \mapsto K(x,t)$ provides an equivalent norm on $\Sigma(X_0, X_1)$ for which it becomes an exact interpolation space.

In the sequel for $t > 0$ let $t\mathbb{R}$ denote the real line $\mathbb{R}$ equipped with the norm $\|x\|_{t\mathbb{R}} = t|x|$. If $\mathcal{F}$ is an exact interpolation functor, its *characteristic function* φ is defined by

$$\varphi(s,t)\mathbb{R} := \mathcal{F}(s\mathbb{R}, t\mathbb{R}).$$

In particular we may work with a function from the class **AC**. For a compatible pair (X_0, X_1) of Banach spaces, we define $(X_0, X_1)_{\varphi,1}$ as the space of all $x \in \Sigma(X_0, X_1)$ for which there exists a sequence $(x_n) \subseteq \Delta(X_0, X_1)$ such that $x = \sum_{n=1}^{\infty} x_n$ in $\Sigma(X_0, X_1)$ and $\sum_{n\geq 1} \varphi(\|x_n\|_0, \|x_n\|_1) < \infty$. Equipped with the norm

$$\|x\|_{\varphi,1} = \inf \Big\{ \sum_{n\geq 1} \varphi(\|x_n\|_0, \|x_n\|_1) : \quad x = \sum_{n=1}^{\infty} x_n \Big\}$$

$(X_0, X_1)_{\varphi,1}$ becomes a Banach space. It can be shown, that the space $(X_0, X_1)_{\varphi,1}$ is an interpolation space. Moreover the interpolation functor $(\cdot,\cdot)_{\varphi,1}$ turns out to be exact and φ is its characteristic function. In addition, the interpolation functor $(\cdot,\cdot)_{\varphi,1}$ possesses a certain minimal property in the following sense. If $\mathcal{F}$ is an arbitrary exact interpolation functor and φ is its characteristic function, then for any Banach couple (X_0, X_1) the following inclusion holds

$$(X_0, X_1)_{\varphi,1} \subseteq \mathcal{F}(X_0, X_1).$$

In addition, the embedding constant is less than one, i.e the above inclusion is a contraction. In the sequel we also consider the interpolation space $(X_0, X_1)_{\varphi,\infty}$ consisting of all $x \in \Sigma(X_0, X_1)$ such that

$$\|x\|_{\varphi,\infty} = \sup_{t>0} \frac{K(t,x)}{\varphi^*(1,t)}$$

is finite. Here $\varphi^* : (0,\infty) \times (0,\infty) \to \mathbb{R}_+$ is given by $\varphi^*(s,t) = (\varphi(1/s, 1/t))^{-1}$. Again, the functor $(\cdot,\cdot)_{\varphi,\infty}$ is exact and φ is its characteristic function. Moreover, it possesses the following maximality property. For an arbitrary exact interpolation functor $\mathcal{F}$ with characteristic function φ and for any interpolation couple (X_0, X_1)

$$\mathcal{F}(X_0, X_1) \subseteq (X_0, X_1)_{\varphi,\infty}$$

and the embedding constant is smaller than one. For more information and proofs we refer the reader to [BK91].
In what follows, let X_0, X_1 be Banach spaces such that $X_0 \hookrightarrow X_1$. Let us deal with sequences given by

$$a_k := \rho(2^{-k}) \quad \text{and} \quad b_k := 2^k \rho(2^{-k}) \quad \text{for } k \in \mathbb{N}.$$

Next, observe the following basic properties

(i) The sequence (a_k) is decreasing.

(ii) The sequence (b_k) is increasing.

(iii) $\rho(\tau) \leq \max\{1, \tau\}$ for any $\tau \in \mathbb{R}_+$.

For $x \in X_0$, define the following expressions.

$$u(x) = \inf\Big\{ \sum_{k=1}^{n} \varphi(\|x_k\|_0, \|x_k\|_1) : \quad n = 1, 2, \ldots;\ x_k \in X_0;\ x = \sum_{k=1}^{n} x_k \Big\}.$$

$$v(x) = \inf\Big\{ \sum_{k=1}^{\infty} \varphi(\|x_k\|_0, \|x_k\|_1) : \quad x_k \in X_0;\ x \overset{X_1}{=} \sum_{k=1}^{\infty} x_k \Big\}.$$

$$w(x) = \inf\Big\{ \max\Big(\sum_{k=1}^{\infty} \|a_k x_k\|_0, \sum_{k=1}^{\infty} \|b_k x_k\|_1 \Big) : \quad x_k \in X_0;\ x \overset{X_1}{=} \sum_{k=1}^{\infty} x_k \Big\}.$$

Observe that any representation $x \overset{X_1}{=} \sum_{k=1}^{\infty} x_k$ with $x_k \in X_0$ such that either $\sum_{k=1}^{\infty} \varphi(\|x_k\|_0, \|x_k\|_1) < \infty$ or $\sum_{k=1}^{\infty} \|a_k x_k\|_0 < \infty$ is absolutely convergent. All above expressions are norms on X_0. The following proposition tells us that these norms are equivalent provided that the function ρ is submultiplicative.

PROPOSITION 6.8. *If $\rho(st) \leq c\rho(s)\rho(t)$ for every $s, t \in \mathbb{R}_+$ then*

$$v(x) \leq u(x) \leq c\, w(x) \leq 2c\, v(x) \text{ for all } x \in X_0.$$

Proof. The inequality $v(x) \leq u(x)$ is trivial.
Let us show that $u(x) \leq c\ w(x)$. For that, assume $x \overset{X_1}{=} \sum_{k=1}^{\infty} x_k$ with $x_k \in X_0$ satisfies $\sum_{k=1}^{\infty} \|a_k x_k\|_0 < \infty$ and $\sum_{k=1}^{\infty} \|b_k x_k\|_1 < \infty$. Given $n \in \mathbb{N}$, define $\widetilde{x}_n = x - \sum_{k=1}^{n} x_k \in X_0$. Since

$$\Big\| a_n \Big(x - \sum_{k=1}^{n} x_k \Big) \Big\|_0 \leq a_n \|x\|_0 + \sum_{k=1}^{n} a_k \|x_k\|_0 \leq a_n \|x\|_0 + \sum_{k=1}^{n} \|a_k x_k\|_0,$$

the sequence $(\|a_n\widetilde{x}_n\|_0)$ is bounded. Since

$$\|b_n\widetilde{x}_n\|_1 \leq b_n \sum_{k=n+1}^{\infty} \|x_k\|_1 \leq \sum_{k=n+1}^{\infty} \|b_k x_k\|_1$$

is a null sequence, we can, for given $\varepsilon > 0$, choose n large enough that

$$\varphi(\|a_n\widetilde{x}_n\|_0, \|b_n\widetilde{x}_n\|_1) < \varepsilon.$$

The submultiplicativity of ρ implies

$$\varphi(s,t) = s\rho\Big(\frac{t}{s}\Big) \leq c\; s\rho(2^{-k})\rho\Big(2^k \frac{t}{s}\Big) = c\; s\; a_k\rho\Big(\frac{b_k t}{a_k s}\Big) = c\;\varphi(a_k s, b_k t)$$

for all $s, t \geq 0$ and $k = 1, 2, \ldots$.
Hence

$$\begin{aligned}
u(x) &\leq \sum_{k=1}^{n} \varphi(\|x_k\|_0, \|x_k\|_1) + \varphi(\|\widetilde{x}_n\|_0, \|\widetilde{x}_n\|_1) \\
&\leq c\sum_{k=1}^{n} \varphi(\|a_k x_k\|_0, \|b_k x_k\|_1) + c\varphi(\|a_n\widetilde{x}_n\|_0, \|b_n\widetilde{x}_n\|_1) \\
&\leq c\sum_{k=1}^{\infty} \varphi(\|a_k x_k\|_0, \|b_k x_k\|_1) + c\varepsilon \leq c\varphi\Big(\sum_{k=1}^{\infty} \|a_k x_k\|_0, \sum_{k=1}^{\infty} \|b_k x_k\|_1\Big) + c\varepsilon \\
&\leq c\max\Big(\sum_{k=1}^{\infty} \|a_k x_k\|_0, \sum_{k=1}^{\infty} \|b_k x_k\|_1\Big) + c\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we obtain

$$u(x) \leq c\max\Big(\sum_{k=1}^{\infty} \|a_k x_k\|_0, \sum_{k=1}^{\infty} \|b_k x_k\|_1\Big).$$

Taking the infimum on the right side yields

$$u(x) \leq c\; w(x).$$

Finally, we have to prove that

$$w(x) \leq 2\; v(x) \text{ for all } x \in X_0.$$

So assume $x \overset{X_1}{=} \sum_{k=1}^\infty x_k$ with $x_k \in X_0$ and $\sum_{k=1}^\infty \varphi(\|x_k\|_0, \|x_k\|_1) < \infty$. As already observed, this implies the convergence of $\sum_{k=1}^\infty \|x_k\|_1$. Let

$$I_n = \{k \in \mathbb{N} : \ 2^n \|x_k\|_1 \leq \|x_k\|_0 \leq 2^{n+1}\|x_k\|_1\}.$$

Since $\|x_k\|_1 \leq \|x_k\|_0$, we have $\bigcup_{n\geq 0} I_n = \mathbb{N}$. Set $y_n = \sum_{k\in I_n} x_k$. Then $x \overset{X_1}{=} \sum_{k=1}^\infty y_k$. Now, it follows that

$$\begin{aligned}\sum_{n=0}^\infty \|a_n y_n\|_0 &\leq \sum_{n=0}^\infty \sum_{k\in I_n} \|\rho(2^{-n})x_k\|_0 = \sum_{n=0}^\infty \sum_{k\in I_n} \varphi(\|x_k\|_0, 2^{-n}\|x_k\|_0) \\ &\leq \sum_{n=0}^\infty \sum_{k\in I_n} \varphi(\|x_k\|_0, 2\|x_k\|_1) \leq 2\sum_{k=1}^\infty \varphi(\|x_k\|_0, \|x_k\|_1).\end{aligned}$$

Consequently we obtain

$$\begin{aligned}\sum_{n=0}^\infty \|b_n y_n\|_1 &\leq \sum_{n=0}^\infty \sum_{k\in I_n} \|2^n\rho(2^{-n})x_k\|_1 = \sum_{n=0}^\infty \sum_{k\in I_n} \varphi(2^n\|x_k\|_1, \|x_k\|_1) \\ &\leq \sum_{n=0}^\infty \sum_{k\in I_n} \varphi(\|x_k\|_0, \|x_k\|_1) = \sum_{k=1}^\infty \varphi(\|x_k\|_0, \|x_k\|_1).\end{aligned}$$

Altogether

$$w(x) \leq \max\left(\sum_{n=0}^\infty \|a_n y_n\|_0, \sum_{n=0}^\infty \|b_n y_n\|_1\right) \leq 2\sum_{k=1}^\infty \varphi(\|x_k\|_0, \|x_k\|_1).$$

Turning to the infimum on the right hand side yields the claim. $\square$

We generalize now a result obtained by U. Matter, see [Mat89, Theorem A].

THEOREM 6.9. *Let $T \in \mathcal{L}(X,Y)$, $S \in \mathcal{L}(X,Z)$ be such that $\|Tx\| \leq \varphi(\|Sx\|, \|x\|)$ holds for all $x \in X$. Let $\ker(S)$ denote the kernel of S. Moreover, let ρ be sub-multiplicative. Then there exists an operator $D : (X/\ker(S), Z)_{\varphi,1} \to Y$ such that the operator T factors as follows: $T = DJ_\varphi Q$, where Q and J_φ denote the canonical quotient map $X \to X/\ker(S)$ and the continuous embedding $X/\ker(S) \hookrightarrow (X/\ker(S), Z)_{\varphi,1}$, respectively. In other words, the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "Q"] & Y \\ X/\ker(S) \arrow[r, "J_\varphi"] & (X/\ker(S), Z)_{\varphi,1} \arrow[u, "D"] \end{tikzcd}$$

Proof. Consider the canonical factorization of S

$$S : X \xrightarrow{Q} X/\ker(S) \xrightarrow{\widetilde{S}} Z.$$

By (6.5) we have $\ker(S) \subset \ker(T)$. This implies that T factors as follows

$$X \xrightarrow{Q} X/\ker(S) \xrightarrow{P} X/\ker(T) \xrightarrow{T_1} Y.$$

We may write $T = \widetilde{T}Q$, where $\widetilde{T} = T_1 P$. Furthermore, from (6.5) we obtain

$$\|\widetilde{T}\widetilde{x}\| \leq \varphi(\|\widetilde{S}\widetilde{x}\|, \|\widetilde{x}\|) \text{ for all } \widetilde{x} \in X/\ker(S).$$

For $\widetilde{x} \in X/\ker(S)$ with $\widetilde{x} = \sum_{k=1}^{n} \widetilde{x}_k$ we have

$$\|\widetilde{T}\widetilde{x}\| \leq \sum_{k=1}^{n} \|\widetilde{T}\widetilde{x}_k\| \leq \sum_{k=1}^{n} \varphi(\|\widetilde{S}\widetilde{x}_k\|, \|\widetilde{x}_k\|).$$

Proposition 6.8 ensures that there exists a constant $C > 0$ such that

$$\|\widetilde{T}\widetilde{x}\| \leq C\|\widetilde{x}\|_{\varphi,1}$$

Define $D_0 : X/\ker(S) \to Y,\ D_0(\widetilde{x}) := \widetilde{T}\widetilde{x}$. By density we may extend the operator D_0 to a continuous map $D : (X/\ker(S), Z)_{\varphi,1} \to Y$. Of course $T = DJ_\varphi Q$. □

6.4 (p, φ)-absolutely continuous operators

Our goal in this section is to investigate some important examples of the interpolative construction introduced in Section 6.2. We begin by reviewing some of the needed results on absolutely p-summing operators. Suppose that $1 \leq p < \infty$. Recall that an operator $T \in \mathcal{L}(X, Y)$ is said to be *absolutely p-summing* or just p-summing, if it takes weak ℓ_p-sequences (x_n) of X (i.e. $(\langle x_n, x' \rangle) \in \ell_p$ for all $x' \in X'$) to strong ℓ_p sequences (Tx_n) of Y (i.e. $(\|Tx_n\|) \in \ell_p$). In fact T is p-summing if and only if there exists a constant C such that for any choice of $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$,

$$\Big(\sum_{i=1}^{n} \|Tx_i\|^p\Big)^{1/p} \leq C \sup_{x' \in B_{X'}} \Big(\sum_{i=1}^{n} |\langle x_i, x' \rangle|\Big)^{1/p}.$$

The least constant C for which the above inequality holds is denoted by $\pi_p(T)$. The class of p-summing operators endowed with the norm π_p constitutes an injective maximal Banach ideal denoted by $[\Pi_p, \pi_p]$. The fundamental characterization of p-summing operators developed by A. Pietsch (see [Pie78]) may be formulated as follows. An operator $T \in \mathcal{L}(X, Y)$ is p-summing if and only if there exist a constant $C \geq 0$ and a regular probability measure μ on $B_{X'}$ such that for each $x \in X$

$$\|Tx\| \leq C\Big(\int_{B_{X'}} |\langle x, x'\rangle|^p d\mu(x')\Big)^{1/p}.$$

By applying the definition of $(\mathcal{A}, \mathcal{B})_\varphi$ to ideals $[\mathcal{A}, \alpha] = [\Pi_p, \pi_p]$ and $[\mathcal{B}, \beta] = [\mathcal{L}, \|\cdot\|]$ we may consider the Banach ideal $[\Pi_{p,\varphi}, \pi_{p,\varphi}] := [(\Pi_p, \mathcal{L})_\varphi, \overline{\gamma}]$ of all (p, φ)-*absolutely continuous operators*. By Theorem 6.5, the ideal norm $\pi_{p,\varphi}$ is equivalent to

$$\widetilde{\pi}_{p,\varphi}(T) = \inf_\lambda\{\varphi(\pi_p(S), \lambda) : \quad \|Tx\| \leq \varphi(\|Sx\|, \lambda\|x\|)\},$$

provided that ρ is submultiplicative. The subsequent result characterizes (p, φ)-absolutely continuous operators by a special factorization property through a suitable space given by the interpolation method $(\cdot, \cdot)_{\varphi,1}$.

THEOREM 6.10. *Let ρ be submultiplicative. For every operator $T \in \mathcal{L}(X, Y)$, the following statements are equivalent*

(i) T is (p, φ)-absolutely continuous.

(ii) There exist a regular probability measure μ on $B_{X'}$ and a constant $C > 0$ such that

$$\|Tx\| \leq \varphi(C\|J_\mu x\|, \|x\|), \quad \textit{for every } x \in X,$$

where $J_\mu : X \to L_p(\mu)$ is the restriction of the canonical map $J_p : C(B_{X'}) \to L_p(\mu)$ and is given by $x \mapsto \langle x, \cdot\rangle$.

(iii) There exist a regular probability measure μ on $B_{X'}$, a constant $C > 0$ and an operator $R : (X/\ker(J_\mu), L_p(\mu))_{\varphi,1} \to Y$ such that $\|R\| \leq C$ and $T = RJ_{\mu,\varphi}Q$. In other words, the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "Q"] & Y \\
X/\ker(J_\mu) \arrow[r, "J_{\mu,\varphi}"] & (X/\ker(J_\mu), L_p(\mu))_{\varphi,1} \arrow[u, "R"]
\end{tikzcd}$$

Proof. The equivalence of (i) and (ii) follows immediately from the definition of (p,φ)-absolutely continuous operators and the Pietsch factorization theorem for p-summing operators. For a proof of the implication (iii) to (i) observe that the continuous embedding $\widetilde{J_\mu} : X/\ker(J_\mu) \to L_p(\mu)$ is p-summing. This fact together with the properties of interpolation norms shows that the embedding $J_{\mu,\varphi} : X/\ker(J_\mu) \to (X/\ker(J_\mu), L_p(\mu))_{\varphi,1}$ is (p,φ)-absolutely continuous. We finally show $(ii) \Rightarrow (iii)$. By assumption there is a probability measure μ on $B_{X'}$ such that

$$\|C^{-1}Tx\| \leq \varphi(\|J_\mu x\|, \|x\|) \quad \text{for all } \ x \in X.$$

Thus by setting $Z = L_p(\mu)$ and $S = J_\mu$ and applying Theorem 6.9, we obtain that $C^{-1}T = DJ_{\mu,\varphi}Q$ for a suitable operator $D : (X/\ker(J_\mu), L_p(\mu))_{\varphi,1} \to Y$. Finally the operator $R = CD$ possesses the desired properties. $\square$

6.5 Applications to approximation quantities and entropy numbers

We start with the introduction of some basic notation for approximation quantities and entropy numbers of linear operators between Banach spaces. Given a closed linear subspace M of X, the inclusion mapping of M into X will be denoted by J_M^X. The k-th *Gelfand number* of $T \in \mathcal{L}(X,Y)$ is given by

$$c_k(T) = \inf\{\|TJ_M^X\| : \ M \subset X \text{ and } \operatorname{codim} M < k\}.$$

The k-th *entropy number* of a bounded set $M \subset X$ is defined as

$$\varepsilon_k(M) = \inf\Big\{\varepsilon > 0 \mid \ \exists x_1, \dots, x_k \in X \text{ such that } M \subset \bigcup_{i=1}^{k} (x_i + \varepsilon B_X)\Big\}.$$

Furthermore the k-th inner entropy number of a bounded set $M \subset X$ is given by

$$\varphi_k(M) = \sup\Big\{\rho > 0 \mid \ \exists x_1, \dots, x_k \in M \text{ such that } \|x_i - x_k\| \geq 2\rho \text{ for } i \neq k\Big\}.$$

For an operator $T \in \mathcal{L}(X,Y)$ between Banach spaces we put

$$\varepsilon_n(T) = \varepsilon_n(T(B_X)) \quad \text{and} \quad \varphi_n(T) = \varphi_n(T(B_X)).$$

Moreover, we study the quantities

$$e_n(T) = \varepsilon_{2^{n-1}}(T) \quad \text{and} \quad f_n(T) = \varphi_{2^{n-1}}(T),$$

called *dyadic entropy numbers* and *inner dyadic entropy numbers*, respectively. For any operator $T \in \mathcal{L}(X, Y)$ we have

$$f_n(T) \leq e_n(t) \leq 2f_n(t). \tag{6.9}$$

The speed of convergence to zero of a sequence of entropy numbers measures "quality" of compactness of the operator under consideration.

Throughout this chapter we use symmetric quasi Banach sequence spaces. By this term we mean a quasi Banach space E consisting of scalar sequences such that $\|(x_n)\|_E = \|(x_n^*)\|_E$. To avoid trivial cases we assume that E contains a sequence with full support. For arbitrary Banach spaces X, Y we define

$$\mathcal{L}_E^{(s)}(X, Y) = \{T \in \mathcal{L}(X, Y) : \quad (s_n(T)) \in E\}$$

with $s = c$ or $s = e$. The above linear space equipped with the quasi-norm defined by

$$\|T|\mathcal{L}_E^{(s)}\| = \|(s_n(T))\|_E,$$

becomes a quasi Banach operator ideal.

More information on entropy numbers and approximation quantities may be found in [Koe86] and [CS90].

LEMMA 6.11. *If $T \in \mathcal{L}(X, Y)$, $S \in \mathcal{L}(X, Z)$ and $R \in \mathcal{L}(X, W)$ are such that (6.5) holds then*

$$c_{n+m-1}(T) \leq \varphi(c_n(S), c_m(R)). \tag{6.10}$$

Proof. For given $\varepsilon > 0$ we may choose subspaces M and N of X such that

$$\|SJ_M^X\| \leq (1 + \varepsilon)c_n(S) \quad \text{and} \quad \operatorname{codim}(M) < n,$$

$$\|RJ_N^X\| \leq (1 + \varepsilon)c_m(R) \quad \text{and} \quad \operatorname{codim}(N) < m.$$

Define $L := M \cap N$. It is easy to verify that

$$\operatorname{codim}(L) \leq \operatorname{codim}(M) + \operatorname{codim}(N) < m + n - 1.$$

Using the above inequality we obtain

$$c_{n+m-1}(T) \le \|TJ_L^X\| = \sup_{\|x\|\le 1} \|(TJ_J^X)x\| \le \sup_{\|x\|\le 1} \varphi(\|(SJ_M^X)x\|, \|(RJ_N^X)x\|)$$

$$\le (1+\varepsilon)\varphi(c_n(S), c_m(R)).$$

Letting $\varepsilon \to 0$ we conclude that (6.10) holds. □

Using a similar argument as in [Pie78] we obtain the following inequality for dyadic entropy numbers.

LEMMA 6.12. *If $T \in \mathcal{L}(X,Y)$, $S \in \mathcal{L}(X,Z)$ and $R \in \mathcal{L}(X,W)$ are such that (6.5) holds then*

$$e_{n+m-1}(T) \le 2\varphi(e_n(S), e_m(R)). \tag{6.11}$$

Proof. Suppose that $\sigma_0 > e_n(S)$ and $\sigma_1 > e_m(R)$. Then we find $z_1, \dots z_{q_0} \in Z$ and $w_1, \dots w_{q_1} \in W$ with

$$S(B_X) \subseteq \bigcup_{h=1}^{q_0} \{z_h + \sigma_0 B_Z\} \ \text{ and } \ R(B_X) \subseteq \bigcup_{h=1}^{q_1} \{w_h + \sigma_1 B_W\}$$

respectively, and $q_0 \le 2^{n-1}, q_1 \le 2^{m-1}$. For given $x_1, \dots, x_p \in B_X$ with $p > 2^{(n+m-1)-1}$ we define

$$I_h := \{i : \ Sx_i \in z_h + \sigma_0 B_Z\}.$$

Since $\sum_{h=1}^{q_0} \text{card}(I_h) \ge p > q_0 q_1$, we have $\text{card}(I_{h_0}) > q_1$ for some h_0. Hence there exist $i, j \in I_{h_0}$ such that Rx_i and Rx_j belong to the same $w_{h_1} + \sigma_1 B_W$. This means that

$$\|Sx_i - Sx_j\| \le 2\sigma_0 \ \text{ and } \ \|Rx_i - Rx_j\| \le 2\sigma_1.$$

Thus we obtain

$$\|Tx_i - Tx_j\| \le 2\varphi(\sigma_0, \sigma_1).$$

By (6.9) we obtain

$$e_{n+m-1}(T) \le 2f_{n+m-1}(T) \le 2\varphi(\sigma_0, \sigma_1).$$

Since this is true for arbitrary $\sigma_0 > e_n(S)$ and $\sigma_1 > e_m(R)$, this completes the proof. □

Let E, E_0, E_1 be quasi normed sequence spaces. The function φ is said to be (E, E_0, E_1)-regular, if for every non-decreasing, positive sequences (s_n) and (t_n) the following inequality holds

$$\big\|(\varphi(s_n, t_n))\big\|_E \leq \varphi\Big(\|(s_n)\|_{E_0}, \|(t_n)\|_{E_1}\Big). \tag{6.12}$$

EXAMPLE: In case when $E = \ell_p, E_0 = \ell_{p_0}$ and $E_1 = \ell_{p_1}$ with $1/p = (1-\theta)/p_0 + \theta/p_1$ and $\varphi(s,t) = s^{1-\theta}t^\theta$ for $0 < \theta < 1$ the above condition becomes the Hölder inequality.

Im summary we can state the following result which generalizes results obtained by H. Jarchow and U. Matter in [JM88].

PROPOSITION 6.13. *Let φ be a (E, E_0, E_1)-regular function. Then*

$$\Big(\mathcal{L}^{(c)}_{E_0}, \mathcal{L}^{(c)}_{E_1}\Big)_\varphi \subseteq \mathcal{L}^{(c)}_E$$

and

$$\Big(\mathcal{L}^{(e)}_{E_0}, \mathcal{L}^{(e)}_{E_1}\Big)_\varphi \subseteq \mathcal{L}^{(e)}_E.$$

Proof. We give the proof only for the case of Gelfand numbers. The entropy number case is similar. Let us assume that φ is a (E, E_0, E_1)-regular function. Then using Lemma 6.11 and Inequality (6.12) we obtain

$$\|(c_n(T))\|_E \leq \|\,(\varphi(c_n(S), c_n(R)))\,\|_E \leq \varphi\big(\|(c_n(S))\|_{E_0}, \|(c_n(R))\|_{E_1}\big).$$

This completes the proof. $\square$

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Lebenslauf

Name:	Mariusz Piotrowski
geboren	am 17. November 1977 in Czarne
Staatsbürgerschaft:	polnisch
Familienstand:	ledig
Abitur:	1996 an der Spezialklasse mathematisch-naturwissenschaftlicher Richtung im LO Szczecinek, Polen.
Studium:	1996–2001: Mathematik und Informatik an der Adam-Mickiewicz-Universität Poznań, Polen. 1999–2000: Stipendiat in Rahmen des Socrates-Erasmus Programms an der BUGH Wuppertal. 2001–2004: wissenschaftlicher Mitarbeiter im Drittmittelprojekt der DFG Hi 584 2-3 an der Friedrich-Schiller-Universität Jena. Betreuer PD Dr. Aicke Hinrichs.
Diplom:	Juli 2001 Mathematik (mit der Gesamtnote sehr gut), März 2001 Informatik

Jena, den 31. August 2004 Mariusz Piotrowski

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Jena, den 31. August 2004 — Mariusz Piotrowski

Vector Valued Fourier Transforms and Absolutely Continuous Operators

vorgelegt von Dipl.-Math. Mariusz Piotrowski

Im Mittelpunkt der Dissertation stehen Untersuchungen zu vektorwertigen Fouriertransformationen und zu absolutstetigen Operatoren. Das Hauptaugenmerk liegt dabei auf den Beziehungen zwischen der Geometrie der Banachräume und den analytischen Eigenschaften der untersuchten Funktionen. Die Argumentation stützt sich wesentlich auf Techniken aus dem Bereich der Operatorenideale und der abstrakten harmonischen Analysis. Die Arbeit gliedert sich in sechs Kapitel, wobei die ersten zwei Kapitel zur Festlegung der Notation und zur Bereitstellung von Hilfsmitteln dienen.

Definitionen zu vektorwertigen Fouriertransformationen

Jede lokalkompakte abelsche Gruppe G (kurz: lka) wird im folgenden zusammen mit einem invarianten *Haar-Maß* μ_G betrachtet. Ein *Charakter* auf G ist ein stetiger Homomorphismus $\gamma : G \to \mathbb{T}$. Die Menge aller Charaktere bildet die sogenannte Dualgruppe G'. Die *Fouriertransformation* $\mathcal{F}_G$ einer Funktion $f \in L_1(G)$ ist definiert durch

$$(\mathcal{F}_G f)(\gamma) = \int_G f(t)\overline{\gamma(t)}\, d\mu_G(t) \quad \text{für} \quad \gamma \in G'.$$

Der Operator $T \in \mathcal{L}(X,Y)$ ist vom *Fourier-Typ* p $(1 \le p \le 2)$ bezüglich G, falls sich der Operator

$$\mathcal{F}_G \otimes T : L_p(G) \otimes X \to L_{p'}(G') \otimes Y$$

zu einem beschränkten linearen Operator von $L_p^X(G)$ nach $L_{p'}^Y(G')$ fortsetzen läßt. Wir fordern also die Gültigkeit der Hausdorff-Young-Ungleichung

$$\|(\mathcal{F}_G \otimes T)f\|_{L_{p'}^Y(G')} \le C\|f\|_{L_p^X(G)}$$

für alle X-wertigen Funktionen $f \in L_p^X(G)$. Hierbei bezeichnet p' den konjugierten Exponent zu p, der durch $1/p + 1/p' = 1$ gegeben ist. Insbesondere hat X Fourier-Typ p, wenn die Identität I_X Fourier-Typ p hat. Die Klasse aller Fourier-Typ p Operatoren versehen mit der Operatorennorm von $\mathcal{F}_G \otimes T$ (sie wird mit $\|T|\mathcal{FT}_p^G\|$ bezeichnet) ist ein Banach Operatorenideal, welches mit $\mathcal{FT}_p^G$ bezeichnet wird. Ein Operator ist vom klassischen Fourier-Typ p, falls er vom Fourier-Typ p bezüglich einer der (und damit aller) Gruppen $\mathbb{Z}, \mathbb{R}, \mathbb{T}$ ist. Die Menge

$$\mathbb{D} = \{x = (x_n)_{n\in\mathbb{N}} : x_n \in \{0,1\}\}$$

versehen mit koordinatenweiser Addition definiert die sogenannte *Cantorgruppe*. Ihr stetiges Analogon ist gegeben durch

$$\mathbb{F} = \{x = (x_n)_{n\in\mathbb{Z}} : x_n \in \{0,1\} \text{ und } x_n \to 0 \text{ für } n \to -\infty\}.$$

Der Banachraum X heißt vom *Rademacher-Typ* p, falls es eine Konstante $C > 0$ gibt, so daß für beliebige $x_1, \dots, x_n \in X$

$$\left(\int_0^1 \Big\| \sum_{k=1}^n r_k(t) x_k \Big\|^2 \, dt\right)^{1/2} \le C \left(\sum_{k=1}^n \|x_k\|^p\right)^{1/p}$$

gilt. Hierbei bezeichnet r_k die k-te Rademacher Funktion. Der Banachraum X ist *B-konvex* genau dann, wenn er einen nichttrivialen ($p > 1$) Rademacher-Typ hat.

Ergebnisse zu vektorwertigen Fouriertransformationen

1. Diskrete und stetige Fouriertransformationen auf der Cantorgruppe

THEOREM 1 *Sei $1 < p < 2$. Für einen Operator $T \in \mathcal{L}(X,Y)$ sind die folgenden Aussagen äquivalent.*

(i) T hat Fourier-Typ p bezüglich der Gruppe $\mathbb{D}$.

(ii) T hat Fourier-Typ p bezüglich der Gruppe $\mathbb{D}^m$ für alle $m \in \mathbb{N}$.

(iii) T hat Fourier-Typ p bezüglich der Gruppe $\mathbb{F}$.

(iv) T hat Fourier-Typ p bezüglich der Gruppe $\mathbb{F}^m$ für alle $m \in \mathbb{N}$.

Außerdem stimmen alle Normen überein.

Der Beweis des Theorems 1 schließt an Ideen von J. Bourgain und H. König an, die das entsprechende Resultat für die klassischen Gruppen $\mathbb{Z}, \mathbb{R}, \mathbb{T}$ bewiesen haben.

2. Das Bourgain-Theorem für unendliche Produkte von zyklischen Gruppen von Primzahlpotenzordnung

THEOREM 2 *Sei m eine Primzahlpotenz. Der Banachraum X ist B-konvex genau dann, wenn er vom nichttrivialen Fourier-Typ p ($1 < p \le 2$) bezüglich der Gruppe $\mathbb{Z}_m^\infty$ ist.*

Dieses Resultat wurde in der Arbeit von J. Bourgain (1988) erwähnt, jedoch nicht bewiesen. In der Literatur waren bisher keine Beweise dieser Aussage bekannt. Darüber hinaus erhalten wir ensprechende Resultate für Klassen von Operatoren. Dabei spielt das Orthonormalsystem von Vilenkin eine wichtige Rolle.

3. Übertragungsprinzipien für Fourier-Typ 2 Operatoren

THEOREM 3 (mit A. Hinrichs) *Jeder Operator vom klassischen Fourier-Typ 2 hat Fourier-Typ 2 bezüglich jeder beliebigen lka Gruppe. Genauer gilt*

$$\mathcal{FT}_2^{\mathbb{T}} \subseteq \mathcal{FT}_2^{G} \quad \textit{und} \quad \|T \,|\, \mathcal{FT}_2^{G}\| \le \|T \,|\, \mathcal{FT}_2^{\mathbb{T}}\|$$

für alle lka Gruppen G und alle $T \in \mathcal{FT}_2^{\mathbb{T}}$.

Das obige Übertragungsprinzip war in einigen Spezialfällen bekannt. Insbesondere, geht dieses Ergebnis im Fall der Cantorgruppe auf A. Hinrichs (2001) züruck. Die Ideen dieser Arbeit konnten erfolgreich auf den Fall von zyklischen Gruppen übertragen werden. Der allgemeine Fall wurde mittels der Strukturtheorie lokalkompakter abelscher Gruppen bewiesen.

Definitionen zu absolutstetigen Operatoren

Im folgenden bezeichnen $\mathcal{A}$ und $\mathcal{B}$ stets Operatorenideale. Darüber hinaus bezeichnen wir mit **AC** die Menge aller stetigen, positiv homogenen Funktionen $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$ mit $\varphi(0,0) = 0, \varphi(s,0) = \varphi(0,t) = 0$ und $\varphi(1,1) = 1$. Wegen der Homogenität von φ schreiben wir oft $\varphi(s,t) = s\rho(t/s)$. Die folgende Definition stellt eine Verallgemeinerung der von U. Matter (1987) eingeführten Prozedur vor. Ein Operator $T \in \mathcal{L}(X,Y)$ liegt in $(\mathcal{A},\mathcal{B})_\varphi$, wenn es Banachräume Z, W und Operatoren $S \in \mathcal{A}(X,Z)$, $R \in \mathcal{B}(X,W)$ gibt, so daß

$$\|Tx\| \leq \varphi(\|Sx\|, \|Rx\|) \text{ für alle } x \in X \tag{1}$$

gilt. Die Lineartität von $(\mathcal{A},\mathcal{B})_\varphi$ sieht man anhand der folgenden Charakterisierung. Ein Operator T liegt in $(\mathcal{A},\mathcal{B})_\varphi$ genau dann, wenn es Banachräume Z_i, W_i und Operatoren $S_i \in \mathcal{A}(X,Z_i), R_i \in \mathcal{B}(X,W_i), i = 1,2,\ldots,n$ gibt, so daß

$$\|Tx\| \leq \sum_{i=1}^{n} \varphi(\|S_i x\|, \|R_i x\|) \text{ für alle } x \in X \tag{2}$$

gilt. Im folgenden seien α und β Idealnormen auf $\mathcal{A}$ bzw. $\mathcal{B}$. Für $T \in (\mathcal{A},\mathcal{B})_\varphi$ definieren wir

$$\gamma(T) = \inf \varphi(\alpha(S), \beta(R)),$$

wobei das Infimum über alle Operatoren S, R genommen wird, so daß die Bedingung (1) gilt. Weiter sei

$$\overline{\gamma}(T) = \inf \sum_{i=1}^{n} \varphi(\alpha(S_i), \beta(R_i)),$$

wobei das Infimum über alle Operatoren S_i, R_i und alle Zahlen $n \in \mathbb{N}$ genommen wird, so daß die Bedingung (2) gilt.
Für ein $t > 0$ bezeichne $t\mathbb{R}$ die Menge der reellen Zahlen $\mathbb{R}$ versehen mit $\|x\|_{t\mathbb{R}} = t|x|$. Für einen exakten Interpolationsfunktor $\mathcal{F}$ ist seine *charakteristische Funktion* φ durch $\varphi(s,t)\mathbb{R} = \mathcal{F}(s\mathbb{R}, t\mathbb{R})$ gegeben. Insbesondere können wir die Funktionen aus der Klasse **AC** betrachten. Für ein Interpolationspaar (X_0, X_1) von Banachräumen definieren wir den Raum $(X_0, X_1)_{\varphi,1}$ als die Menge aller $x \in X_0 + X_1$, für welche es eine Folge $(x_n) \subseteq X_0 \cap X_1$ gibt, so daß $x = \sum_{n=1}^{\infty} x_n$ in $X_0 + X_1$ und $\sum_{n\geq 1} \varphi(\|x_n\|_0, \|x_n\|_1) < \infty$ gilt. Versehen mit der Norm

$$\|x\|_{\varphi,1} = \inf \left\{ \sum_{n\geq 1} \varphi(\|x_n\|_0, \|x_n\|_1) : \quad x = \sum_{n=1}^{\infty} x_n \right\}$$

wird $(X_0, X_1)_{\varphi,1}$ zu einem Banachraum. Man kann zeigen, daß $(X_0, X_1)_{\varphi,1}$ ein Interpolationsraum ist. Die Gelfandzahlen c_k und dyadischen Entropiezahlen e_k von $T \in \mathcal{L}(X,Y)$ sind gegeben durch

$$c_k(T) = \inf\{\|TJ_M^X\| : \quad M \subset X \text{ und } \operatorname{codim} M < k\} \quad \text{bzw.}$$

$$e_k(T) = \inf\left\{\varepsilon > 0 \mid \; \exists y_1, \ldots, y_{2^{k-1}} \in Y, \;\text{ so daß } T(B_X) \subset \bigcup_{i=1}^{2^{k-1}} (y_i + \varepsilon B_Y)\right\}.$$

Ergebnisse zu absolutstetigen Operatoren

4. Äquivalente Norm

THEOREM 4 *Sei ρ eine submultiplikative Funktion, d.h. es gibt eine Konstante $c > 0$, so daß $\rho(st) \leq c\rho(s)\rho(t)$ für alle $s, t \in \mathbb{R}_+$ gilt. Dann gilt*

$$\overline{\gamma}(T) \leq \gamma(T) \leq c\,\overline{\gamma}(T).$$

Ist die Funktion ρ also submultiplikativ, so sind die Abbildungen γ und $\overline{\gamma}$ äquivalent.

5. Zusammenstellung der Eigenschaften von $(\mathcal{A}, \mathcal{B})_\varphi$

THEOREM 5 *Seien $\varphi \in \boldsymbol{AC}$ und $\rho : [0, \infty) \to \mathbb{R}_+$ gegeben durch $\varphi(s,t) = s\rho(t/s)$. Seien ausserdem $\mathcal{A}$ und $\mathcal{B}$ Operatorenideale. Dann ist $(\mathcal{A}, \mathcal{B})_\varphi$ ein Operatorenideal. Falls zusätzlich $(\mathcal{A}, \alpha)$, $(\mathcal{B}, \beta)$ quasinormierte Banachideale sind und ρ eine submultiplikative Funktion ist, dann ist $[(\mathcal{A}, \mathcal{B})_\varphi, \overline{\gamma}]$ ein Banachideal.*

6. Faktorisierung durch Interpolationsräume und (p, φ)-absolutstetige Operatoren.

THEOREM 6 *Sei ρ submultiplikativ. Für jeden Operator $T \in \mathcal{L}(X,Y)$ sind die folgenden Bedingungen äquivalent*

(i) T *ist (p,φ)-absolutstetig ($T \in (\Pi_p, \mathcal{L})_\varphi(X,Y)$).*

(ii) Es existieren ein reguläres Wahrscheinlichkeitsmaß μ auf $B_{X'}$ und eine Konstante $C > 0$, so daß

$$\|Tx\| \leq \varphi(C\|J_\mu x\|, \|x\|), \quad \text{für alle } x \in X$$

gilt, wobei $J_\mu : X \to L_p(\mu)$ gegeben ist durch $x \mapsto \langle x, \cdot \rangle$.

(iii) Es existieren ein Wahrscheinlichkeitsmaß μ auf $B_{X'}$ und ein Operator $R : (X/\ker(J_\mu), L_p(\mu))_{\varphi,1} \to Y$, so daß $T = RJ_{\mu,\varphi}Q$. Das folgende Diagramm kommutiert also:

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "Q"] & Y \\
X/\ker(J_\mu) \arrow[r, "J_{\mu,\varphi}"] & (X/\ker(J_\mu), L_p(\mu))_{\varphi,1} \arrow[u, "R"']
\end{tikzcd}$$

7. Abschätzungen von Approximationsgrössen

THEOREM 7 *Genügen $T \in \mathcal{L}(X,Y)$, $S \in \mathcal{L}(X,Z)$ und $R \in \mathcal{L}(X,W)$ der Ungleichung (1), so gilt*

$$c_{n+m-1}(T) \leq \varphi(c_n(S), c_m(R)),$$
$$e_{n+m-1}(T) \leq 2\varphi(e_n(S), e_m(R)).$$